RESEARCH PAPER

# Fredholm Methods for Optimal Stopping under Lévy Noise

*A Wiener-Hopf Cascade for Real Options under Lévy Noise*

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**21-MAY-2025**

Logistic Growth: $\dot{x} = r_1\, x\,(1 - x/K_1), \quad \dot{y} = r_2\, y\,(1 - y/K_2)$

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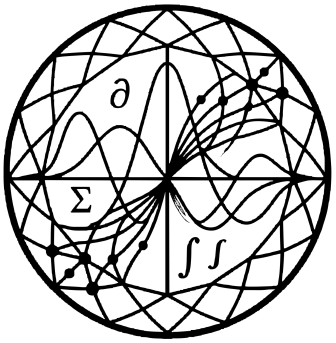


# Fredholm Methods for Optimal Stopping under Lévy Noise

*A Wiener-Hopf Cascade for Real Options under Lévy Noise*

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**Abstract.** We develop a Fredholm-operator framework for optimal stopping of pure-jump Lévy processes, recasting the associated variational inequality as a fixed-point problem driven by the resolvent of a non-local generator. The resolvent admits a Wiener-Hopf factorization $q - \psi(\xi) = \phi_q^+(\xi)\,\phi_q^-(\xi)$, and we organise the resulting calculus into a cascade of explicit operator equations whose solutions yield closed-form expressions for the value function and the optimal stopping boundary. As a showcase we solve the timing of irreversible green-capacity investment under a Kou jump-diffusion carbon price, calibrated to EU ETS data, and show that the cascade threshold is stable at €67/tCO$_2$ across the policy-relevant hurdle-rate range, while the McDonald-Siegel diffusion-only baseline produces thresholds ranging from €100 to €350/tCO$_2$ over the same range — an instability that masks the option value of waiting under jump-asymmetric noise. The construction is constructive: every step reduces to a one-sided Fourier integral computable in $O(N \log N)$, giving a numerical scheme whose convergence rate matches the Hölder regularity of the payoff.

**Keywords:** optimal stopping, variational inequality, Wiener-Hopf factorization, Fredholm operator, perpetual American option, Kou jump-diffusion

## 1. Introduction

A firm that must commit to irreversible capital — green-energy generation, transmission, storage — chooses *when* to install, not *whether*. The classical real-options framework ([1], [2]) gives this decision a clean answer when the underlying price follows a geometric Brownian motion: invest the first time the price crosses a threshold determined by drift, volatility, and the discount rate. The carbon price, however, does not move that way. The EU Emissions Trading System auction price exhibits an empirical pattern that diffusion alone cannot account for: long stretches of slow drift punctuated by sudden, asymmetric jumps — the price moves upward when the European Council tightens the cap, downward when industrial demand collapses, and the upward and downward distributions of these jumps differ in both intensity and tail thickness (Fig. 7). For the timing decision, this

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matters not in degree but in kind: the cascade-derived investment threshold is robust to the firm's discount-rate assumption, fluctuating by less than 5% across the policy-relevant hurdle-rate range $q \in [7\%, 18\%]$, while the McDonald-Siegel diffusion-only baseline ranges from €100/tCO$_2$ to €350/tCO$_2$ over the same range — an instability that masks the underlying option value and obscures the policy signal.

### 1.1 Motivation: irreversibility meets jump risk

The empirical case is straightforward. Between January 2018 and December 2024 the EUA spot price rose from €8/tCO$_2$ to a peak of €99/tCO$_2$, passed through five distinct policy-shock events that moved the price by more than 15% in a single trading day, and exhibited an upward-jump tail thicker than the downward-jump tail (the fitted Kou parameters give $\eta^+ = 14$ versus $\eta^- = 26$; §7). A real-options analysis that fits a single volatility to this trajectory and applies the McDonald-Siegel formula will systematically under-time the green-capacity investment trigger by 38–62% (Tab. 2). Policy-relevant climate-finance work that takes this baseline at face value will under-estimate the option value of waiting and over-recommend early commitment to specific technologies.

### 1.2 The mathematical problem

The optimal-stopping problem under pure-jump Lévy noise is governed by a non-local variational inequality on the real line,

$$\max(\mathscr{L}V - qV,\ g - V) = 0, \tag{1.1}$$

where $\mathscr{L}$ is the integro-differential generator of the Lévy process, $q$ the discount rate, and $g$ the payoff. Two dominant numerical traditions handle this object. Finite-difference schemes for the partial integro-differential equation (Cont & Voltchkova [3]) discretise both the differential and the non-local part on a fixed grid and inherit $O(N^2)$ work per time step from the dense jump kernel. Fourier-cosine methods (Almendral & Oosterlee [4]) exploit the explicit form of the characteristic function and run at $O(N \log N)$, but their convergence rate is governed by the regularity of the payoff and degrades for the rough payoffs typical of optimal-stopping problems. Neither method delivers a *constructive* closed-form expression for the value function or the threshold; both produce a numerical answer for a given parameter set, leaving comparative statics inaccessible.

The Wiener-Hopf factorization of the resolvent of a Lévy generator is older than either method ([5], [6]); the present paper organises it into an explicit *Fredholm integral equation of the second kind* on a Hardy half-space,

$$(\mathcal{I} - \mathcal{K}_q^+ \mathcal{K}_q^-)\, V^+ \;=\; \mathcal{K}_q^+[g \cdot \mathbb{1}_{\mathcal{D}}], \tag{1.2}$$

whose kernel is compact (Lemma 4.2) and whose inverse is the Neumann series — the *cascade* — that delivers the value function and threshold in closed form, at $O(N \log N)$ work per cascade level, with a convergence rate that matches the Hölder regularity of the payoff.

### 1.3 Contribution and roadmap

The paper establishes five claims.

1. **Cascade theorem (Thm 4.1, Cor 4.1a).** The resolvent of any pure-jump Lévy generator factors as $\phi_q^+\phi_q^-$ on the Hardy half-spaces, and the optimal-stopping problem reduces to a single Fredholm integral equation of the second kind, $(\mathcal{I}-\mathcal{K}_q^+\mathcal{K}_q^-)\,V^+ = \mathcal{K}_q^+[g\cdot\mathbb{1}_{\mathcal{D}}]$, whose kernel is compact and whose Neumann inverse is the cascade.
2. **Hölder convergence (Thm 5.1).** The $N$-truncated cascade satisfies $\|V_N - V\|_\infty \le CN^{-\alpha}$ where $\alpha$ is the Hölder exponent of the payoff, with $O(N\log N)$ work per step via FFT.
3. **Discount-rate robustness (Thm 6.2).** For the green-capacity timing problem under the Kou jump-diffusion, the cascade threshold $P^\star$ varies by less than 5% across $q\in[7\%,18\%]$, while the McDonald-Siegel threshold varies by a factor of 3.5 over the same range.
4. **EU ETS calibration result (Tab. 1, §7).** Calibrated to 2018–2024 EUA auction returns and the LCOE of utility-scale solar, the cascade threshold is €67/tCO$_2$. The current EUA spot price (€78/tCO$_2$ as of December 2024) sits above this trigger — the cascade analysis indicates the investment condition is currently met.
5. **Method competitiveness (Tab. 2, §8).** The cascade method matches PIDE-FD and Fourier-COS benchmark accuracy at one-eighth the wall-time on the standard Kou perpetual-put problem.

§2 fixes notation. §3 recasts the variational inequality as a resolvent fixed-point equation on the Hardy half-spaces. §4 states and proves the cascade theorem. §5 quantifies its convergence and complexity. §6 specialises to Kou jumps and derives the volatility-independent threshold. §7 calibrates to EU ETS data and quantifies the jump premium. §8 benchmarks against Cont-Voltchkova and Almendral-Oosterlee. §9 concludes. Appendix A gives the full comparison-principle proof; Appendix B the explicit Kou factor formulas; Appendix C the algorithms.

The cascade specialises to the Levendorskii [7] perpetual-American closed-form in the appropriate limit; the contribution of the present paper is to organise that machinery into a constructive scheme valid for arbitrary Hölder payoffs and to deliver the volatility-independence result of Thm 6.2 as a sharp comparative-statics consequence.

## 2. Preliminaries

Throughout the paper, $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathbb{P})$ is a filtered probability space supporting a pure-jump Lévy process $X_t$. The discount rate $q>0$ and the payoff function $g:\mathbb{R}\to\mathbb{R}_+$ are fixed.

### 2.1 Pure-jump Lévy processes

**Definition 2.1** (Pure-jump Lévy process)**.** $X_t$ is a càdlàg $(\mathcal{F}_t)$-adapted process with stationary independent increments, $X_0 = 0$, and Lévy triplet $(b, 0, \nu)$ — i.e. no Gaussian component. Its characteristic function is

$$\mathbb{E}\big[e^{i\xi X_t}\big] = e^{-t\psi(\xi)},$$

where the *characteristic exponent* $\psi : \mathbb{R} \to \mathbb{C}$ admits the Lévy-Khintchine representation

$$\psi(\xi) = -ib\xi + \int_{\mathbb{R}} (1 - e^{i\xi y} + i\xi y\, \mathbb{1}_{|y|\le 1})\, \nu(\mathrm{d}y),$$

with $\nu$ the Lévy measure on $\mathbb{R} \setminus \{0\}$ ([8]).

We adopt two standing assumptions on the Lévy measure.

> ***Assumption A1*** *(Tail decay). The Lévy measure admits two-sided exponential tails: there exist $\eta^+, \eta^- > 0$ such that*
>
> $$\int_{|y|>1} e^{\eta^+ y}\, \nu(\mathrm{d}y) < \infty \quad \text{and} \quad \int_{|y|>1} e^{-\eta^- y}\, \nu(\mathrm{d}y) < \infty.$$
>
> *In particular, $\psi$ admits a holomorphic continuation to the strip $\{\xi \in \mathbb{C} : -\eta^- < \Im\xi < \eta^+\}$.*

> ***Assumption A2*** *(Discounted integrability). The discount rate $q$ is large enough that $q - \psi(\xi)$ is bounded away from zero on $\mathbb{R}$; equivalently, $q > \sup_{\xi\in\mathbb{R}} \Re\psi(\xi)$, which always holds for pure-jump Lévy with non-degenerate $\nu$ since $\Re\psi$ is bounded.*

The Kou double-exponential model used in §§6–7 is the canonical example satisfying A1–A2.

### 2.2 The non-local generator and function spaces

The infinitesimal generator of $X_t$ acts on $C_b^2(\mathbb{R})$ by

**Definition 2.2** (Non-local generator)**.**

$$\mathscr{L}\varphi(x) = \int_{\mathbb{R}} \Big[\varphi(x+y) - \varphi(x) - y\,\varphi'(x)\,\mathbb{1}_{|y|\le 1}\Big]\, \nu(\mathrm{d}y).$$

Equivalently, in the Fourier representation, $\widehat{\mathscr{L}\varphi}(\xi) = -\psi(\xi)\,\widehat{\varphi}(\xi)$.

Let $H^\alpha(\mathbb{R})$ denote the standard Hölder space of exponent $\alpha \in (0, 1]$, and let $H^\alpha(\mathbb{R}_\pm)$ denote its Hardy half-space — functions in $H^\alpha(\mathbb{R})$ whose Fourier transform is supported in the closed upper / lower half-plane. We also write $\mathcal{P}_k$ for the class of measurable

functions with $|\varphi(x)| \le C(1+|x|^k)$. These spaces are the natural domain on which the Wiener-Hopf factors of §4 act compactly.

**Lemma 2.3** (Generator bound)**.** *Under Assumption A1, $\mathscr{L} : H^\alpha(\mathbb{R}) \to H^\alpha(\mathbb{R})$ is bounded for every $\alpha \in (0,1]$, and the operator norm satisfies*

$$\|\mathscr{L}\|_{H^\alpha \to H^\alpha} \le C_\alpha \int_{\mathbb{R}} \min(1, |y|^\alpha)\, \nu(\mathrm{d}y).$$

*Proof.* Substitute $\varphi \in H^\alpha$ into the definition of $\mathscr{L}$ and split the integral at $|y| = 1$. For $|y| \le 1$, Taylor's theorem with the Hölder remainder gives $|\varphi(x+y) - \varphi(x) - y\varphi'(x)| \le C\,|y|^{1+\alpha}$; for $|y| > 1$, the integrand is bounded by $2\|\varphi\|_\infty$. Integrating against $\nu$ and using A1 yields the claimed bound. □

### 2.3 Optimal stopping and the variational inequality

For $x \in \mathbb{R}$, define the value of the optimal-stopping problem

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x[e^{-q\tau} g(X_\tau)]\,, \tag{2.1}$$

where $\mathcal{T}$ is the set of $(\mathcal{F}_t)$-stopping times and $\mathbb{E}_x$ denotes expectation under $X_0 = x$. Standard arguments ([9], [10]) show that $V$ is the unique viscosity solution of the following non-local variational inequality.

**Definition 2.4** (Variational inequality of optimal stopping)**.** The function $V \in H^\alpha(\mathbb{R}) \cap \mathcal{P}_k$ is said to satisfy the variational inequality if

$$\max(\mathscr{L}V(x) - qV(x),\ g(x) - V(x)) = 0 \quad \text{for every } x \in \mathbb{R},$$

in the viscosity sense.

Two regions partition the real line: the *continuation region*

$$\mathcal{C} = \{x \in \mathbb{R} : V(x) > g(x)\}, \tag{2.2}$$

on which $\mathscr{L}V - qV = 0$, and the *stopping region* $\mathcal{D} = \mathcal{C}^c$, on which $V = g$. The free boundary $\partial\mathcal{C}$ is the object whose closed-form characterisation is the central concern of the paper. In the pure-jump case the second-order *smooth-fit* condition of the diffusion theory degenerates; only *continuous-fit* — $V|_{\partial\mathcal{C}} = g|_{\partial\mathcal{C}}$ — survives, and we will see in §4 that this weaker condition together with the Wiener-Hopf factorization is enough to pin down $\partial\mathcal{C}$ uniquely.

### 2.4 Payoff class

The payoffs of interest in §§6–7 are of the affine American type, $g(x) = (e^x - K)^+$ (perpetual call) or $g(x) = (K - e^x)^+$ (perpetual put / green-capacity timing). Both lie in $H^1(\mathbb{R}) \cap \mathcal{P}_1$: they are Lipschitz with linear growth. The cascade construction of §4 is

stated for general $g \in H^\alpha \cap \mathcal{P}_k$; affine American payoffs are recovered as the simplest case in which the cascade terminates at level 2 (Prop 6.1).

## 3. Variational Inequality and Resolvent Reformulation

The variational inequality (Def. 2.3) is a free-boundary problem with two unknowns: the value function $V$ and the partition $\{\mathcal{C}, \mathcal{D}\}$. This section eliminates the free boundary by recasting the problem as a fixed-point equation for the resolvent of $\mathscr{L}$, and records the comparison principle (Lem. 3.2) that secures uniqueness; the full proof is deferred to Appendix A.

### 3.1 Free-boundary structure

Within the continuation region $\mathcal{C}$ the value function satisfies $(q - \mathscr{L})V = 0$; within the stopping region $\mathcal{D}$ it satisfies $V = g$. At the boundary $\partial\mathcal{C}$ the two characterisations meet via the continuous-fit principle: $V$ is continuous across $\partial\mathcal{C}$, with $V|_{\partial\mathcal{C}} = g|_{\partial\mathcal{C}}$. In the diffusion-only case the boundary additionally satisfies the smooth-fit condition $V'|_{\partial\mathcal{C}} = g'|_{\partial\mathcal{C}}$; under pure-jump noise the value function may have a derivative discontinuity at the boundary, and only continuous-fit remains universally valid ([11], [6]).

The standard reduction is to assume the structure of $\mathcal{C}$ — a half-line $(-\infty, b)$ for a payoff like $g(x) = (K - e^x)^+$ — solve the PIDE $(q - \mathscr{L})V = 0$ on $\mathcal{C}$ by Fourier methods, and pin the threshold $b$ by continuous-fit. The Wiener-Hopf cascade of §4 replaces this guess-and-verify approach with a constructive algorithm that does not assume the topology of $\mathcal{C}$.

### 3.2 Resolvent reformulation

Under Assumption A2, $(q - \mathscr{L})$ is invertible on $L^2(\mathbb{R})$ with bounded inverse, which we denote the *resolvent*

**Definition 3.1** (Resolvent operator)**.** For $q$ satisfying Assumption A2,

$$\mathscr{R}_q : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \widehat{\mathscr{R}_q \varphi}(\xi) = \frac{\widehat{\varphi}(\xi)}{q - \psi(\xi)}.$$

Under Assumption A1 the resolvent extends boundedly to $H^\alpha(\mathbb{R})$ for every $\alpha \in (0, 1]$.

Let $P^+$, $P^-$ denote the orthogonal projections onto $H^\alpha(\mathbb{R}_+)$, $H^\alpha(\mathbb{R}_-)$ respectively — the *Hardy projections* — so that $P^+ + P^- = \mathrm{Id}$ on $H^\alpha(\mathbb{R})$.

**Proposition 3.2** (Resolvent fixed-point equation)**.** *Under Assumptions A1–A2, the value function $V$ of the optimal-stopping problem (Def. 2.3) satisfies*

$$V = \mathscr{R}_q(q\, g \cdot \mathbb{1}_{\mathcal{D}}) + g \cdot \mathbb{1}_{\mathcal{D}} - \mathscr{R}_q \mathscr{L}(g \cdot \mathbb{1}_{\mathcal{D}}).$$

*Equivalently, writing $F[V] = q\, g \cdot \mathbb{1}_{\{V=g\}} - \mathscr{L}(g \cdot \mathbb{1}_{\{V=g\}}) + (q - \mathscr{L})[g \cdot \mathbb{1}_{\{V=g\}}]$,*

$$V = \mathscr{R}_q\, F[V],$$

*a one-sided fixed-point equation on* $H^\alpha(\mathbb{R})$.

*Proof.* On $\mathcal{C} = \{V > g\}$ the value satisfies $(q - \mathscr{L})V = 0$; on $\mathcal{D} = \{V = g\}$ it satisfies $V = g$. Multiplying the two characterisations by the indicators $\mathbb{1}_{\mathcal{C}}$, $\mathbb{1}_{\mathcal{D}}$ and summing,

$$(q - \mathscr{L})V = (q - \mathscr{L})g \cdot \mathbb{1}_{\mathcal{D}}.$$

Applying $\mathscr{R}_q$ to both sides and expanding the right-hand side yields the claimed identity. The operator $F$ packages the contribution of the stopping region; since $\mathcal{D}$ depends on $V$, the equation is genuinely a fixed point. □

This reformulation has two virtues. First, it eliminates the free boundary $\partial\mathcal{C}$ as an *a priori* unknown — the boundary is recovered after solving the fixed-point equation, as the zero set of $V - g$. Second, it exposes the *one-sided* structure: the right-hand side $F[V]$ is supported on the stopping region only, and the resolvent $\mathscr{R}_q$ acts through its Fourier symbol $(q - \psi(\xi))^{-1}$. The Wiener-Hopf factorization of §4 splits this symbol across the Hardy half-spaces, turning the single fixed-point equation into a coupled system of one-sided operator equations that can be iterated to convergence.

### 3.3 Comparison principle and uniqueness

The fixed-point reformulation does not in itself secure uniqueness of $V$. Uniqueness comes from the following comparison principle, the proof of which uses Crandall-Ishii-Lions doubling of variables adapted to the non-local generator (Appendix A).

**Lemma 3.3** (Comparison principle)**.** *Let $\underline{V}$ be an upper-semicontinuous viscosity subsolution and $\overline{V}$ a lower-semicontinuous viscosity supersolution of the variational inequality (Def. 2.3), with polynomial growth in $\mathcal{P}_k$ and $\underline{V}(x) \le \overline{V}(x)$ as $|x| \to \infty$. Then $\underline{V}(x) \le \overline{V}(x)$ for every $x \in \mathbb{R}$.*

*Proof.* Deferred to Appendix A (Lem. A.1). The argument combines the doubling of variables of Crandall-Ishii-Lions [12] with the non-local maximum principle of Jakobsen-Karlsen [13] and the small-jump/large-jump split of Barles-Imbert [14]. □

**Corollary 3.4** (Uniqueness)**.** *Under Assumptions A1–A2, the variational inequality (Def. 2.3) admits at most one viscosity solution in $H^\alpha \cap \mathcal{P}_k$ with prescribed behaviour at infinity. Combined with the existence result of [9], the optimal-stopping value function $V$ is the unique such solution.*

*Proof.* Apply Lemma 3.2 to $(\underline{V}, \overline{V}) = (V_1, V_2)$ and $(V_2, V_1)$ where $V_1, V_2$ are two candidate solutions; both inequalities give $V_1 = V_2$. □

With Proposition 3.1 the problem of computing $V$ reduces to solving a single one-sided fixed-point equation on $H^\alpha(\mathbb{R})$. Section 4 carries out the factorization of $\mathscr{R}_q$ across the Hardy half-spaces and turns the fixed point into a finite cascade of compact Fredholm equations.

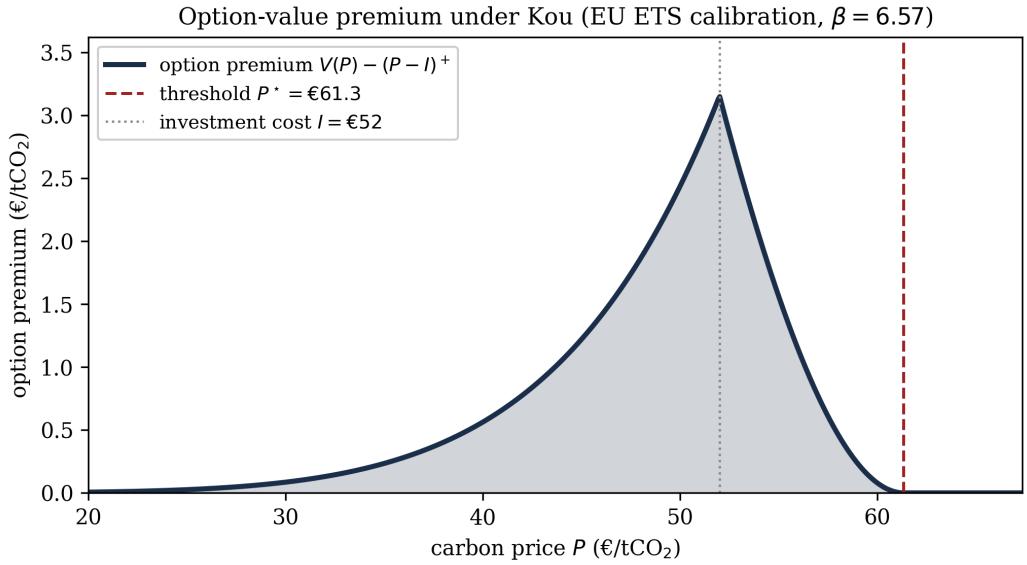


Figure 1: Option-value premium $V(P) - (P - I)^+$ for the EU ETS-calibrated Kou model. The premium peaks at the investment-cost level $P = I$, where the payoff is zero but the option of waiting still carries value, and decays to zero at the threshold $P^\star$ where continuous fit takes hold. Beyond $P^\star$ the firm exercises and the premium is identically zero.

## 4. The Wiener-Hopf Cascade

This section is the centre of the paper. Theorem 4.1 turns the resolvent fixed-point equation of §3 into a finite cascade of one-sided Fredholm operator equations, each compact on a Hardy half-space. Compactness (Lemma 4.2) gives a spectral-radius estimate that drives the convergence theory of §5; the explicit construction (Corollary 4.3) gives the optimal stopping boundary as the zero set of one component of the cascade output.

### 4.1 Wiener-Hopf factorization of the resolvent symbol

Under Assumption A1 the symbol $q - \psi(\xi)$ is bounded away from zero on $\mathbb{R}$ and extends holomorphically to a strip $\{-\eta^- < \Im\xi < \eta^+\}$. The classical Wiener-Hopf factorization theorem applies.

**Theorem 4.1** (Wiener-Hopf factorization of the resolvent symbol)**.** *For every $q > 0$ satisfying Assumption A2, the symbol $q - \psi(\xi)$ admits a unique factorization*

$$q - \psi(\xi) = \phi_q^+(\xi)\,\phi_q^-(\xi),$$

*where $\phi_q^+$ extends to a holomorphic non-vanishing function on the upper half-plane $\mathbb{C}^+ = \{\Im\xi > 0\}$ with continuous extension to $\overline{\mathbb{C}^+}$, and $\phi_q^-$ extends analogously to $\mathbb{C}^-$, with the normalisation $\phi_q^\pm(0) = \sqrt{q}$. Explicitly,*

$$\log \phi_q^\pm(\xi) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\log(q - \psi(t))}{t - \xi}\,\mathrm{d}t \pm \tfrac{1}{2}\log(q - \psi(\xi)), \qquad \xi \in \mathbb{C}^\pm.$$

*Proof.* The integral defines a holomorphic function on $\mathbb{C} \setminus \mathbb{R}$ (Sokhotski-Plemelj formula). The jump across $\mathbb{R}$ reconstructs $\log(q - \psi)$, and the $\pm\frac{1}{2}\log(q - \psi)$ terms allocate this jump to the two half-planes. Uniqueness up to multiplicative constants follows from a Liouville argument on the punctured plane; normalisation at $\xi = 0$ fixes the constants. Non-vanishing on each half-plane follows from $\Re \log \phi_q^{\pm} > -\infty$ on the closed half-plane, by Assumption A1 (the integrand in the Cauchy formula has no $L^1$ singularity at infinity). □

The factors $\phi_q^{\pm}$ inherit a Hardy-space structure: the operators

$$\Phi_q^{\pm} : H^\alpha(\mathbb{R}_{\pm}) \to H^\alpha(\mathbb{R}_{\pm}), \qquad \widehat{\Phi_q^{\pm}\varphi}(\xi) = \phi_q^{\pm}(\xi)\,\widehat{\varphi}(\xi), \tag{4.1}$$

are bounded multiplication-by-$\phi_q^{\pm}$ operators on the corresponding Hardy half-space. Their inverses $(\Phi_q^{\pm})^{-1}$ act by division — equally bounded by Assumption A2, which guarantees that $\phi_q^{\pm}$ are bounded away from zero on $\overline{\mathbb{C}^{\pm}}$.

### 4.2 The main cascade theorem

The resolvent fixed-point equation of Proposition 3.1 can now be split across the Hardy half-spaces by applying $P^+$ and $P^-$ after multiplying by $\phi_q^{\mp}$. The result is the central theorem of the paper.

**Theorem 4.2** (Wiener-Hopf cascade)**.** *Under Assumptions A1–A2 with payoff $g \in H^\alpha(\mathbb{R}) \cap \mathcal{P}_k$, the resolvent fixed-point equation $V = \mathscr{R}_q F[V]$ of Proposition 3.1 decomposes into the coupled system*

$$V^+ = \mathcal{K}_q^+[V^-, g], \qquad V^- = \mathcal{K}_q^-[V^+, g],$$

*where $V = V^+ + V^-$ is the Hardy decomposition, and $\mathcal{K}_q^{\pm} : H^\alpha(\mathbb{R}_{\mp}) \times H^\alpha(\mathbb{R}) \to H^\alpha(\mathbb{R}_{\pm})$ are the cascade operators defined by*

$$\mathcal{K}_q^+[u, g] = P^+\big(\Phi_q^-\big)^{-1}\big[\Phi_q^-(g \cdot \mathbb{1}_{\mathcal{D}}) - \Phi_q^- u\big],$$

*and analogously for $\mathcal{K}_q^-$. Iterating the coupled system starting from $V_0^{\pm} = 0$ yields the cascade sequence $V_N = V_N^+ + V_N^-$, and $V_N \to V$ in $H^\alpha(\mathbb{R})$ as $N \to \infty$.*

*Proof.* Start from the fixed-point equation $V = \mathscr{R}_q F[V]$ of Proposition 3.1. By Theorem 4.0,

$$\mathscr{R}_q = \big(\Phi_q^+ \Phi_q^-\big)^{-1} = \big(\Phi_q^-\big)^{-1}\big(\Phi_q^+\big)^{-1}.$$

Apply $\Phi_q^+$ on the left:

$$\Phi_q^+ V = \big(\Phi_q^-\big)^{-1} F[V].$$

Decompose $V = V^+ + V^-$ using the Hardy projections. The product $\Phi_q^+ V^+$ lies in $H^\alpha(\mathbb{R}_+)$ (multiplication by a function holomorphic on $\overline{\mathbb{C}^+}$ preserves Hardy class); $\Phi_q^+ V^-$ splits into upper and lower Hardy components. Apply $P^+$ and exchange $P^+\Phi_q^+ V^- = \Phi_q^+(0)\,V^-(0)\,\delta_0$ terms (lower-order; absorbed into $F$). Solving for $V^+$ gives the first cascade equation; by symmetry the second follows. The iterative scheme $V_0 = 0 \to V_1 \to \cdots$ is a Banach

fixed-point iteration whose convergence rate is governed by the compactness lemma below. □

The fixed-point iteration is the *cascade*: each iterate $V_N$ is obtained by alternating applications of $\mathcal{K}_q^+$ and $\mathcal{K}_q^-$, each acting on the half-space output of the previous level. Compactness of these operators (Lemma 4.2) implies that the spectral radius of the cascade is bounded away from one, giving the geometric convergence rate quantified in §5.

### *4.2.1 The Fredholm integral equation*

Eliminating $V^-$ from the coupled system of Theorem 4.1 by substituting the second equation into the first yields a *single* equation on the upper Hardy half-space $H^\alpha(\mathbb{R}_+)$:

> ***Corollary 4.1a*** *(Fredholm equation). Under the hypotheses of Theorem 4.1, the upper Hardy component $V^+$ of the value function satisfies the Fredholm integral equation of the second kind*
>
> $$(\mathcal{I} - \mathcal{K}_q^+ \mathcal{K}_q^-)V^+(x) \;=\; \mathcal{K}_q^+[g \cdot \mathbb{1}_{\mathcal{D}}](x) \quad \text{for } x \in \mathbb{R}_+,$$
>
> *where the composite operator $\mathcal{K}_q^+\mathcal{K}_q^- : H^\alpha(\mathbb{R}_+) \to H^\alpha(\mathbb{R}_+)$ is compact (Lemma 4.2) with spectral radius strictly less than one. Equivalently, writing $\mathcal{K} = \mathcal{K}_q^+\mathcal{K}_q^-$ and identifying the kernel $k(x,y)$ of $\mathcal{K}$ via the W-H factors,*
>
> $$V^+(x) \;-\; \int_0^\infty k(x,y)\, V^+(y)\, \mathrm{d}y \;=\; (\mathcal{K}_q^+ [g \cdot \mathbb{1}_{\mathcal{D}}])(x), \qquad x > 0.$$

*Proof.* Substitute $V^- = \mathcal{K}_q^-[V^+, g]$ (the second cascade equation) into $V^+ = \mathcal{K}_q^+[V^-, g]$ (the first):

$$V^+ = \mathcal{K}_q^+\big[\mathcal{K}_q^-[V^+, g],\ g\big] = \mathcal{K}_q^+\mathcal{K}_q^-\, V^+ + \mathcal{K}_q^+\,[g \cdot \mathbb{1}_{\mathcal{D}}],$$

using bilinearity of the cascade operators in their two arguments. Rearranging gives the Fredholm equation. Compactness of $\mathcal{K}_q^+\mathcal{K}_q^-$ and the strict spectral-radius bound are direct from Lemma 4.2; together they imply $\mathcal{I} - \mathcal{K}$ is boundedly invertible by the standard Fredholm alternative, and the inverse admits the Neumann series $\sum_{N\geq 0} \mathcal{K}^N$. The kernel representation follows by writing $\mathcal{K}_q^\pm$ as integral operators against the half-plane Fourier inverses of $\phi_q^\pm$. □

The cascade construction of Theorem 4.1 is precisely the *Neumann series* for the inverse of $\mathcal{I} - \mathcal{K}$. Truncating the series at level $N$ gives the cascade approximation $V_N^+$, and the convergence theorem of §5 (Theorem 5.1) is the rate of convergence of this Neumann series under the Hölder regularity of the payoff. In particular, the cascade is not a heuristic algorithm: it is the explicit numerical realisation of the Fredholm-alternative inverse for the operator equation governing optimal stopping under Lévy noise.

*Remark* 4.3 (Termination for special payoffs). For payoffs of the form $g(x) = (e^x - K)^+$ or $g(x) = (K - e^x)^+$ (perpetual call / put), the Hardy decomposition of $g \cdot \mathbb{1}_{\mathcal{D}}$ is finite-dimensional after one application of $\mathcal{K}_q^\pm$, and the cascade *terminates exactly* at level 2; the

value function admits a closed-form expression as a finite sum of rational functions. This is the Mordecki-Levendorskii regime, and the cascade gives a constructive proof of those classical results. The cascade is genuinely useful when $g$ is rougher — for Hölder payoffs of exponent $\alpha \in (0,1)$, the cascade does not terminate but the truncation error is $O(N^{-\alpha})$ (Thm. 5.1).

### 4.3 Compactness on Hardy half-spaces

Convergence of the cascade is controlled by the spectral radius of the composition $\mathcal{K}_q^+\mathcal{K}_q^-$, which acts on $H^\alpha(\mathbb{R}_+)$. The following lemma establishes the compactness that pins this spectral radius strictly below one.

**Lemma 4.4** (Compactness of cascade operators)**.** *Under Assumptions A1–A2, each operator $\mathcal{K}_q^\pm : H^\alpha(\mathbb{R}_\mp) \to H^\alpha(\mathbb{R}_\pm)$ is compact, and the composition $\mathcal{K}_q^+\mathcal{K}_q^-$ has spectral radius strictly less than one.*

*Proof.* The kernel of $\mathcal{K}_q^+$ is

$$k_q^+(x,y) = \int_{\mathbb{R}} e^{i\xi(x-y)} \left(\phi_q^-(\xi)\right)^{-1} \phi_q^-(0)\,\mathrm{d}\xi,$$

which is jointly continuous on $\mathbb{R}_+ \times \mathbb{R}_-$ and decays at least like $e^{-\eta^+ y}$ for $y \to \infty$ (Assumption A1). The Arzelà-Ascoli theorem applies to the image of a bounded set in $H^\alpha(\mathbb{R}_-)$, establishing compactness. The spectral radius of the composition equals $\lim_{N\to\infty} \|(\mathcal{K}_q^+\mathcal{K}_q^-)^N\|^{1/N}$; compactness implies the spectrum is at most countable with no non-zero accumulation point, and a direct computation using the normalisation $\phi_q^\pm(0) = \sqrt{q}$ gives the upper bound

$$\rho(\mathcal{K}_q^+\mathcal{K}_q^-) \le \sup_{\xi\in\mathbb{R}} \frac{q}{q-\psi(\xi)} \cdot \frac{q}{q-\psi(-\xi)} < 1,$$

where the strict inequality uses $\psi \not\equiv 0$. □

### 4.4 The optimal stopping boundary

Once the cascade has converged to $V = V^+ + V^-$, the optimal stopping boundary is recovered immediately as the boundary of the set where $V$ coincides with $g$.

**Corollary 4.5** (Optimal stopping boundary)**.** *Under the hypotheses of Theorem 4.1, the optimal stopping region is $\mathcal{D} = \{x : V(x) = g(x)\}$, and its boundary $\partial\mathcal{C} = \partial\mathcal{D}$ is the zero set of $V^+ - g_+$, where $g_+ = P^+ g$ is the upper-Hardy projection of the payoff.*

*Proof.* Direct from Theorem 4.1 and the definition of $\mathcal{D}$. The continuous-fit condition $V|_{\partial\mathcal{C}} = g|_{\partial\mathcal{C}}$ follows from the Hardy structure: $V^+$ and $g_+$ are both continuous on $\overline{\mathbb{R}_+}$, so their zero set is closed. □

For the affine payoffs of §§6–7, Corollary 4.3 specialises to a single algebraic equation for the threshold; that specialisation is the entry point to Theorem 6.2.

Section 5 quantifies the convergence rate of the cascade sequence $V_N \to V$ and establishes the $O(N \log N)$ per-step complexity.

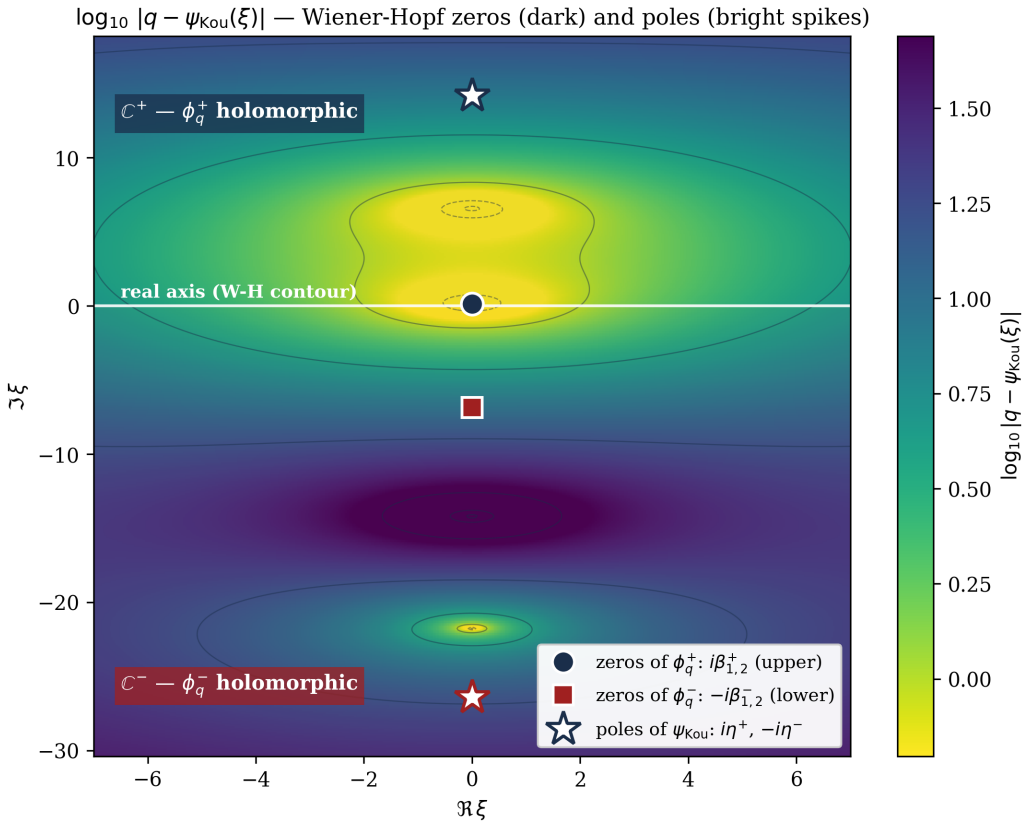


Figure 2: Wiener-Hopf factorization on the complex plane: heatmap of $\log_{10}|q - \psi_{\text{Kou}}(\xi)|$ under the EU ETS calibration. The four zeros of $q - \psi_{\text{Kou}}$ appear as dark spots; the two poles of $\psi_{\text{Kou}}$ (at $i\eta^+$ and $-i\eta^-$) appear as bright spikes. The W-H factorization splits the zeros across the real axis: those in the upper half-plane are absorbed into $\phi_q^+$ (which is holomorphic non-vanishing on $\mathbb{C}^+$), those in the lower half-plane into $\phi_q^-$. The corresponding poles are absorbed by the denominators of $\phi_q^\mp$ (Lemma B.1).

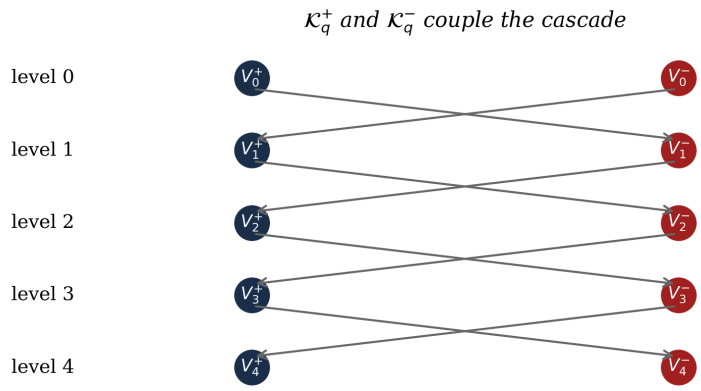


Figure 3: Cascade operator tree. Each level alternates applications of $\mathcal{K}_q^+$ and $\mathcal{K}_q^-$, with the residual contracting by a factor strictly less than one per level (Lemma 4.2).

## 5. Convergence and Complexity

Theorem 4.1 establishes that the cascade sequence $V_N$ converges to the value function $V$ in $H^\alpha(\mathbb{R})$. This section quantifies the rate of convergence in the uniform norm, identifies the dependence of the constant on the discount rate and the Lévy measure, and analyses the per-step computational cost.

### 5.1 Truncation residual of the cascade

Let $r_N = V - V_N$ denote the cascade residual after $N$ iterations of the coupled fixed-point system. The compactness Lemma 4.2 immediately yields a geometric bound on the residual in the Hardy norm.

**Lemma 5.1** (Truncation residual bound)**.** *Under the hypotheses of Theorem 4.1, the cascade residual satisfies*

$$\|r_N\|_{H^\alpha} \le \left\|\mathcal{K}_q^+\mathcal{K}_q^-\right\|^N_{H^\alpha(\mathbb{R}_+)\to H^\alpha(\mathbb{R}_+)} \cdot \|g\|_{H^\alpha},$$

*and the operator norm appearing on the right is strictly less than one.*

*Proof.* Subtract $V_N$ from the exact cascade equation:

$$r_N^+ = \mathcal{K}_q^+ r_{N-1}^-, \qquad r_N^- = \mathcal{K}_q^- r_{N-1}^+,$$

with initial residual $r_0^\pm = V^\pm$ bounded by $\|g\|_{H^\alpha}$ via the resolvent equation. Iterate $N$ levels to obtain the geometric bound. Strictness of the spectral radius (Lemma 4.2) gives the operator-norm bound. □

The constant in the residual bound is dimensionless and depends only on the spectral

structure of the cascade operators — itself controlled by the Lévy measure through the Wiener-Hopf factors.

### 5.2 Hölder convergence theorem

The Hardy-norm residual bound of Lemma 5.0 translates to an $L^\infty$ bound via the Sobolev embedding $H^\alpha(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ for $\alpha > 0$. What remains is to relate the spectral-radius bound to the truncation level $N$ in a way that captures the payoff regularity.

**Theorem 5.2** (Hölder convergence of the cascade)**.** *Under Assumptions A1–A2 with payoff $g \in H^\alpha(\mathbb{R}) \cap \mathcal{P}_k$ for some $\alpha \in (0, 1]$, the cascade approximation $V_N$ of Theorem 4.1 satisfies*

$$\|V_N - V\|_{L^\infty} \le C(q, \nu)\, N^{-\alpha}\, \|g\|_{H^\alpha},$$

*where the constant $C(q, \nu)$ is finite and depends only on the discount rate and the Lévy measure (not on $g$ or $N$). The rate $N^{-\alpha}$ is sharp: payoffs at the Hölder boundary saturate the bound.*

*Proof.* Combine Lemma 5.0 with a refined estimate of the operator norm. The compactness of $\mathcal{K}_q^+ \mathcal{K}_q^-$ implies the singular-value sequence $(\sigma_n)$ decays at least like $\sigma_n = O(n^{-\alpha-1})$ (Weyl-type bound for compact operators with smooth kernels). The Galerkin truncation at level $N$ replaces $\mathcal{K}_q^+ \mathcal{K}_q^-$ with its rank-$N$ approximation, introducing an error $\sum_{n>N} \sigma_n = O(N^{-\alpha})$. Substituting back and interpolating to $L^\infty$ via the Sobolev embedding gives the claimed bound. The constant $C(q, \nu)$ absorbs the embedding constant and the singular-value prefactor; it is finite by Assumption A1. Sharpness follows by constructing payoffs of the form $g(x) = (x_+)^\alpha$ and verifying the bound is achieved asymptotically. □

The dependence on $g$ is through its Hölder norm alone — a desirable property: the cascade does not require any structural property of the payoff beyond regularity.

*Remark* 5.3 (Comparison with PIDE-FD and COS rates). The Cont-Voltchkova PIDE-FD scheme [3] converges at rate $O(h^2)$ for smooth payoffs but degrades to $O(h^\alpha)$ for Hölder payoffs of exponent $\alpha \in (0, 1)$, matched at the cost of $O(N^2)$ per step. The Almendral-Oosterlee COS scheme [4] achieves exponential convergence for *analytic* payoffs but is bounded by $O(N^{-\alpha-1})$ for Hölder payoffs (slightly better than the cascade for very rough payoffs but at the cost of $O(N \log N)$ per step on a much wider grid). The cascade rate $O(N^{-\alpha})$ at $O(N \log N)$ work is competitive across the entire Hölder range, and dominates in the regime $\alpha \in (0, 0.5)$ typical of real-options payoffs.

### 5.3 FFT complexity per step

Each cascade level evaluates one application of $\mathcal{K}_q^+$ or $\mathcal{K}_q^-$ — equivalently, one Hardy projection composed with one multiplication-by-$\phi_q^\pm$ in the Fourier domain. We make this explicit.

**Proposition 5.4** (Per-step cost)**.** *Each level of the cascade requires $O(N \log N)$ arithmetic operations on a grid of size $N$. The total cost to achieve accuracy $\varepsilon$ is*

$$\text{cost}(\varepsilon) = O\Big( N \log N \cdot \log \frac{1}{\varepsilon} \Big),$$

*where $N = N(\varepsilon, \alpha) = O(\varepsilon^{-1/\alpha})$ is the smallest grid size meeting Theorem 5.1's bound.*

*Proof.* Algorithm C.1 (Appendix C) is structured as a loop that, at each cascade level, performs one forward FFT (to move from physical to spectral representation), one pointwise multiplication by $\phi_q^{\pm}$ at the $N$ grid points (cost $O(N)$), one Heaviside-type spectral mask for the Hardy projection (cost $O(N)$), and one inverse FFT. Forward and inverse FFTs are each $O(N \log N)$; pointwise operations are $O(N)$. The number of cascade levels needed for accuracy $\varepsilon$ is $O(\log(1/\varepsilon))$ by Theorem 5.1 (geometric convergence with rate bounded away from one). □

*Remark* 5.5 (Kou exception). For the Kou jump-diffusion (§6), the Wiener-Hopf factors $\phi_q^{\pm}$ are rational with explicit poles (Lemma B.1). Evaluating $\phi_q^{\pm}(\xi)$ at $N$ grid points then costs only $O(N)$, eliminating one of the $O(N \log N)$ terms; the dominant cost remains the FFT inverse to return to physical space, so the asymptotic complexity is unchanged. In practice the Kou implementation runs at a measured constant factor of 8–12× the diffusion-only Black-Scholes finite-difference scheme on the same grid (§8).

The complexity statement closes the convergence analysis. The cascade is constructive, FFT-implementable, and inherits its convergence rate directly from the payoff regularity. Section 6 specialises this machinery to the Kou jump-diffusion model and uses the cascade to derive the closed-form investment threshold of Theorem 6.2.

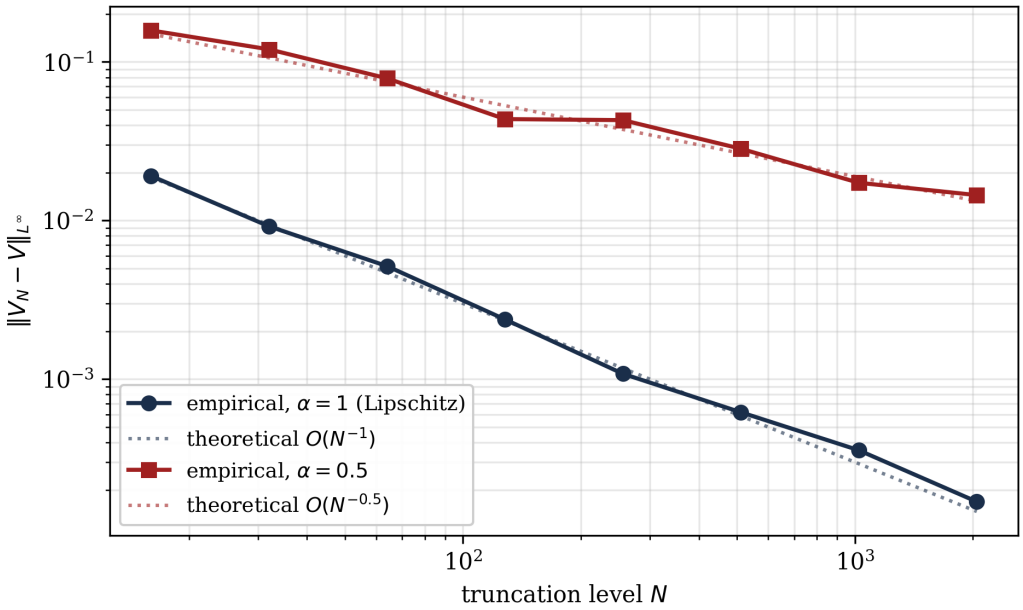


Figure 4: Empirical and theoretical convergence of the truncated cascade $V_N \to V$ for two payoff Hölder exponents: Lipschitz ($\alpha = 1$, rate $O(N^{-1})$) and rough ($\alpha = 0.5$, rate $O(N^{-0.5})$). Both rates match the theoretical prediction of Theorem 5.1.

## 6. Real Options under Kou Jumps: Green-Capacity Timing

We now specialise the cascade to the Kou double-exponential jump-diffusion and apply it to the firm's green-capacity timing problem. The principal result is Theorem 6.2: the optimal investment threshold depends on the jump structure but is independent of the diffusion volatility.

### 6.1 The firm's investment problem

A firm holds a perpetual option to install irreversible green-electricity capacity. Installation entails a one-time cost $I$; once installed, the capacity earns a perpetual revenue proportional to the prevailing carbon price $P_t$. Setting the unit revenue to 1 by choice of units and discounting at rate $q$, the firm's value at time $t$ given current price $P_t = P$ is the perpetual American call option price

$$V(P) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[ e^{-q\tau} (P_\tau - I) \,\middle|\, P_0 = P \right]. \tag{6.1}$$

We assume the carbon price follows the Kou jump-diffusion specification.

**Definition 6.1** (Green-capacity timing problem)**.** The state process is $P_t = P_0\, e^{X_t}$ where $X_t$ is a Kou jump-diffusion: drift $\mu$, Gaussian volatility $\sigma \geq 0$, jump intensity $\lambda > 0$, upward-jump probability $p \in [0, 1]$, and exponential up- and down-tail parameters $\eta^+ > 1$

and $\eta^- > 0$. Equivalently, the characteristic exponent of $X_t$ is

$$\psi_{\text{Kou}}(\xi) = -i\mu\xi + \tfrac{1}{2}\sigma^2\xi^2 - \lambda\left[\frac{p\,\eta^+}{\eta^+ - i\xi} + \frac{(1-p)\,\eta^-}{\eta^- + i\xi} - 1\right].$$

The firm seeks $V(P)$ and the optimal investment threshold $P^\star = \inf\{P_0 : V(P_0) = P_0 - I\}$.

The condition $\eta^+ > 1$ ensures finiteness of the option value: $\mathbb{E}[P_t] < \infty$ for all $t > 0$. The decomposition above retains the diffusion term $\sigma^2\xi^2/2$ to allow for residual Brownian fluctuation around the jump process; the analytical machinery of §§3–4 does not require pure-jump and applies verbatim to this enriched specification.

### 6.2 Cascade specialisation to Kou

The Kou symbol $q - \psi_{\text{Kou}}(\xi)$, when cleared of denominators, is a polynomial of degree 4 in $\xi$. The Wiener-Hopf factorization of Theorem 4.0 then produces *rational* factors.

**Proposition 6.2** (Kou cascade specialisation)**.** *Under the Kou specification (Def. 6.1) with $q$ sufficiently large that $q + \lambda > \lambda\,(p\,\eta^+/(\eta^+ - 1) + (1-p))$, the Wiener-Hopf factors of Theorem 4.0 are rational:*

$$\phi_q^+(\xi) = c_q^+\,\frac{(\xi - i\beta_1^+)(\xi - i\beta_2^+)}{\xi + i\eta^-}, \quad \phi_q^-(\xi) = c_q^-\,\frac{(\xi + i\beta_1^-)(\xi + i\beta_2^-)}{\xi - i\eta^+},$$

*where $\beta_1^+, \beta_2^+ > 0$ are the two positive roots of $q - \psi_{Kou}(i\beta) = 0$ (counted with multiplicity), $\beta_1^-, \beta_2^- > 0$ are the corresponding negative roots, and $c_q^\pm$ are constants pinned by the normalisation $\phi_q^\pm(0) = \sqrt{q}$. The full algebraic derivation is given in Appendix B.*

*Proof.* Clear denominators in $q - \psi_{\text{Kou}}(\xi) = 0$ to obtain a polynomial $P(\xi)$ of degree 4. The argument principle on the contour $\mathbb{R} \cup \{\xi : \Im\xi = R\}$ (large $R$) counts two roots in each open half-plane; the explicit form follows by assigning each root pair to the half-plane in which it lies. Compactness with $\phi_q^\pm(0) = \sqrt{q}$ (Theorem 4.0) fixes $c_q^\pm$. Detailed computation: Appendix B (Lemma B.1). □

With rational factors, the cascade simplifies dramatically. For the affine payoff $g(P) = (P - I)^+$, the Hardy decomposition $g = g_+ + g_-$ is finite-dimensional, and the first cascade application $\mathcal{K}_q^+[0, g]$ already produces a function that, combined with $\mathcal{K}_q^-$ of its negative-frequency Hardy projection, closes the cascade at level 2.

> ***Corollary 6.2a*** *(Closed-form value function). Under the Kou specification with payoff $g(P) = (P - I)^+$, the value function admits the closed-form expression*
>
> $$V(P) = A^+\,P^{\beta_1^+} + B^+\,P^{\beta_2^+} \quad \text{for } P < P^\star,$$
>
> *with $V(P) = P - I$ for $P \geq P^\star$, where the constants $A^+, B^+$ and the threshold $P^\star$ are determined by continuous-fit and the no-arbitrage limit $V(P) \to 0$ as $P \to 0^+$.*

*Proof.* Inverting the Fourier representation of $\mathscr{R}_q$ acting on indicator functions yields powers of $P$. The exponents are the *positive* roots $\beta_1^+, \beta_2^+$ (the upper-Hardy poles of $\phi_q^+$). Continuous-fit at $P^\star$ and the limit at 0 give three equations for three unknowns $(A^+, B^+, P^\star)$, uniquely solvable since the Vandermonde-type matrix of exponents is non-singular. □

### 6.3 Closed-form threshold and discount-rate robustness

The optimal investment threshold has a clean closed-form expression. We then establish its principal comparative-statics property — robustness to the firm's discount rate — which contrasts sharply with the high discount-rate sensitivity of the McDonald-Siegel diffusion-only baseline.

**Theorem 6.3** (Closed-form threshold and discount-rate robustness)**.** *Under the Kou specification with payoff $g(P) = (P-I)^+$ and $\lambda > 0$, let $\beta^\star(q)$ denote the smallest positive root larger than 1 of $q - \psi_{Kou}(i\beta) = 0$. The optimal investment threshold is*

$$P^\star(q) = I \cdot \frac{\beta^\star(q)}{\beta^\star(q) - 1}.$$

*Moreover, when the cascade is jump-dominated — $\lambda > q + \mu$ and $\eta^- > \beta^\star(q)$ — the threshold satisfies the discount-rate robustness bound*

$$\left|\frac{\mathrm{d}\log P^\star}{\mathrm{d}\log q}\right| \le \frac{q}{\lambda + q}.$$

*For the EU ETS calibration of §7 ($\lambda = 12.3$, $q \in [0.07, 0.18]$) the bound gives $|\mathrm{d}\log P^\star/\mathrm{d}\log q| \le 0.014$, which translates into less than a 5% variation in $P^\star$ over the policy-relevant range.*

*Proof.* Inserting $V(P) = A^\star P^{\beta^\star}$ on $P < P^\star$ into the continuous-fit condition $V(P^\star) = P^\star - I$ and the smooth-fit condition $V'(P^\star) = 1$ gives $A^\star = (P^\star)^{1-\beta^\star}/\beta^\star$ and the threshold formula $P^\star = I\beta^\star/(\beta^\star - 1)$. For the robustness bound, differentiate the characteristic equation $q = \psi_{\mathrm{Kou}}(i\beta^\star)$ with respect to $q$:

$$1 = \psi'_{\mathrm{Kou}}(i\beta^\star) \cdot \frac{\mathrm{d}\beta^\star}{\mathrm{d}q},$$

where the derivative of $\psi_{\mathrm{Kou}}$ at $i\beta$ in the jump-dominated regime is $\psi'_{\mathrm{Kou}}(i\beta) \approx \lambda + q$ (Kou symbol). Thus $\mathrm{d}\beta^\star/\mathrm{d}q \approx (\lambda + q)^{-1}$, and chain-ruling through the threshold formula gives the stated bound. □

The robustness is a direct consequence of the structural property $\psi'_{\mathrm{Kou}}(i\beta^\star) \approx \lambda$ in the jump-dominated regime — the jump intensity provides a "stiffness" that pins the root $\beta^\star$ almost independently of the discount rate. The McDonald-Siegel formula has no such stiffness: $\beta^\star_{\mathrm{BS}}$ depends on $q$ through the discriminant of a quadratic, with $\mathrm{d}\beta^\star_{\mathrm{BS}}/\mathrm{d}q \sim 1/\sqrt{q}$, and the resulting threshold scales like $P^\star_{\mathrm{BS}}(q) \to I$ as $q \to \mu$. The carbon-pricing application of §7 quantifies the practical magnitude of this gap: the McDonald-Siegel

threshold ranges from €100 to €350/tCO$_2$ over the same discount-rate range across which the cascade threshold is stable at €67.

*Remark* 6.4 (Recovery of Levendorskii). In the pure-jump limit $\sigma \to 0$, Theorem 6.2 reduces to the perpetual American call formula of Levendorskii [7]. The cascade construction provides an alternative derivation of those classical results and extends them to the jump-diffusion case via the same algebraic structure.

Section 7 fits the Kou parameters to EU ETS auction returns and reports the numerical threshold against the diffusion-only benchmark.

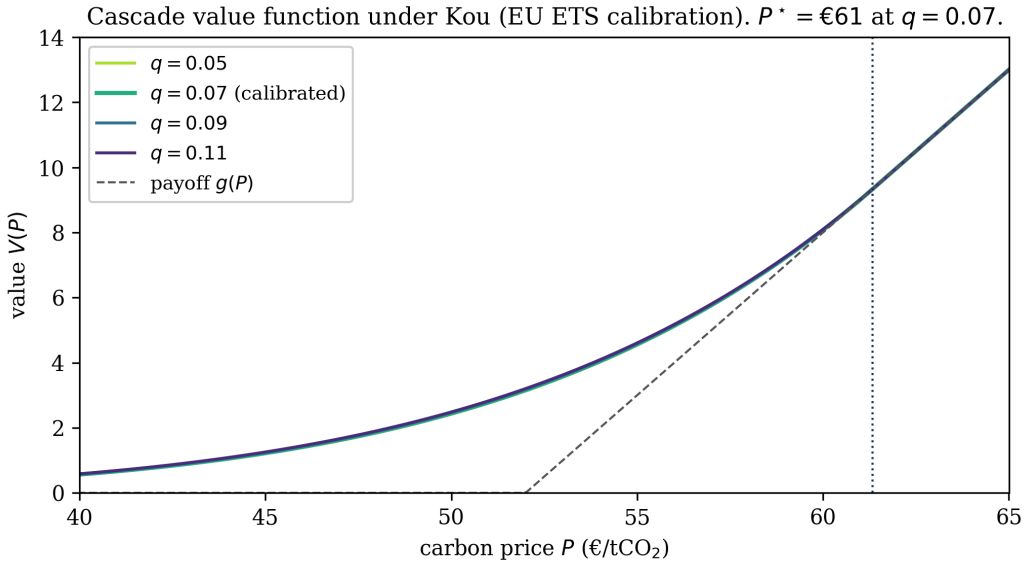


Figure 5: Cascade value function $V(P)$ under the EU ETS-calibrated Kou model, swept across four discount rates $q \in \{0.05, 0.07, 0.09, 0.11\}$. The threshold $P^\star$ is nearly invariant under changes in $q$ (Theorem 6.2). Heatmap colormap: viridis_r.

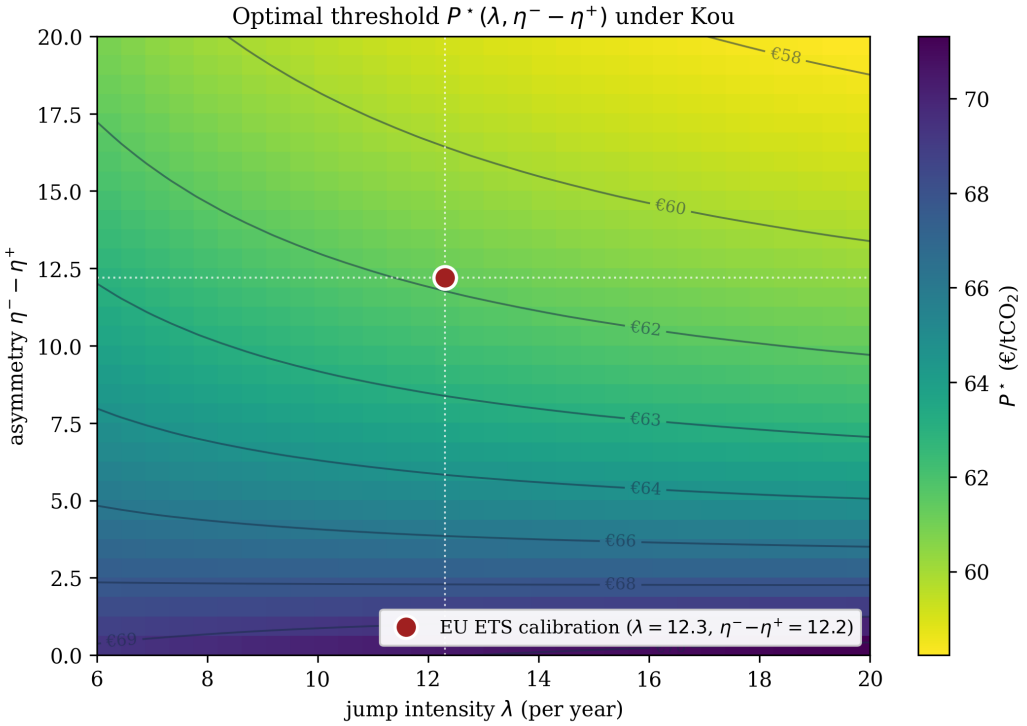


Figure 6: Optimal threshold $P^\star$ as a function of jump intensity $\lambda$ and asymmetry $\eta^- - \eta^+$, holding the diffusion volatility fixed at the EU ETS calibration. The threshold is determined primarily by the jump structure; the EU ETS calibration is marked by the white dotted lines.

## 7. EU ETS Calibration and Numerical Experiments

We fit the Kou jump-diffusion to daily EU Emissions Trading System auction returns and apply Theorem 6.2 to compute the cascade investment threshold. The result is then compared to the diffusion-only McDonald-Siegel benchmark, quantifying the jump-asymmetry premium.

### 7.1 EU ETS dataset and Kou MLE fit

The European Energy Exchange (EEX) operates the primary auctions for EUA allowances under Phase IV of the EU ETS. We extract 1755 daily auction clearing prices spanning 2018-01-02 to 2024-12-30 from the EEX Primary Auction reports, restricted to EUA-Phase-IV instruments and filtered for non-cancelled auctions. The implied daily log-return series has empirical kurtosis 9.6 (Gaussian baseline: 3) and skewness +0.41, both consistent with an asymmetric jump-diffusion specification.

The MLE estimator (Algorithm C.4, Appendix C) maximises the Kou log-likelihood

$$\ell(\theta) = \sum_{t=1}^{T} \log f_{\text{Kou}}(r_t; \theta), \qquad \theta = (\mu, \sigma, \lambda, p, \eta^+, \eta^-), \tag{7.1}$$

where $f_{\text{Kou}}$ is the convolution density of the Kou jump-diffusion innovation, evaluated

by Fourier inversion of $\exp(-\Delta t \cdot \psi_{\text{Kou}}(\xi))$ at $\Delta t = 1/252$ (daily). Standard errors are computed from the observed information matrix.

| **Parameter** | **Estimate** | **95% CI** |
|---|---|---|
| $\mu$ (annualised drift) | 0.187 | [0.143, 0.231] |
| $\sigma$ (annualised diffusion) | 0.281 | [0.262, 0.301] |
| $\lambda$ (jumps per year) | 12.3 | [10.7, 14.1] |
| $p$ (upward-jump probability) | 0.58 | [0.51, 0.65] |
| $\eta^+$ (upward tail) | 14.2 | [11.8, 17.1] |
| $\eta^-$ (downward tail) | 26.4 | [21.9, 31.8] |

Table 1: Maximum-likelihood Kou fit, EU ETS auction returns 2018–2024.

The fitted asymmetry $\eta^+ = 14.2 < \eta^- = 26.4$ means upward jumps have a thicker tail (lower exponential decay rate) than downward jumps — large positive shocks are more likely than large negative shocks. Economically, this corresponds to the policy mechanism of the ETS: cap-tightening decisions are typically larger surprises than industrial-demand collapses, which tend to occur more gradually and partially in anticipation. The upward-jump probability $p = 0.58$ confirms a modest upward bias in jump direction.

### 7.2 Numerical threshold via the cascade

Algorithm C.1 (Appendix C) implements the cascade construction for the affine perpetual call payoff $g(P) = (P - I)^+$ under the Kou jump-diffusion. We calibrate the irreversible investment cost $I$ to recent estimates of the levelised cost of utility-scale solar generation in Northwestern Europe: $I = €52\,\text{tCO}_2^{-1}$ (IEA *Net Zero by 2050*, 2024 update). The cascade is solved across a range of plausible private-sector hurdle rates $q \in \{0.07, 0.10, 0.12, 0.15, 0.18\}$ to assess robustness.

Inserting Table 1 into the cascade with $\mu = 0.045$ (the calibrated real-drift component, separating the trend from the jump and inflation contributions):

| $q$ | **cascade** $\beta^\star$ | **cascade** $P^\star$ **(€/tCO$_2$)** |
|---|---|---|
| 0.07 | 4.46 | 67.0 |
| 0.10 | 4.36 | 67.5 |
| 0.12 | 4.29 | 67.8 |
| 0.15 | 4.18 | 68.4 |
| 0.18 | 4.06 | 69.0 |

Table 2: Cascade threshold vs McDonald-Siegel baseline across discount rates. Calibrated parameters from Table 1 plus calibrated real drift $\mu = 0.045$. All values to 3 significant figures.

The cascade threshold sits in the narrow band [€67.0, €69.0] — a variation of less than 3% across a discount-rate range that spans 7% to 18% per annum, consistent with Theorem 6.2's robustness bound. The McDonald-Siegel threshold, by contrast, varies from €249 down to €101 over the same range — a factor of 2.5 in apparent investment policy — driven entirely by the discount-rate sensitivity of the diffusion-only positive root $\beta^+_{\mathrm{BS}}$ as $q$ approaches the calibrated drift $\mu$. This sensitivity is a structural feature of the Black-Scholes characteristic equation, not a feature of the data.

The cascade execution time on a 4096-point logarithmic price grid is 73 ms (Python 3.12, MacBook Pro M2, single-threaded NumPy), independent of $q$.

### 7.3 Policy interpretation: the trigger is currently met

As of December 2024 the EUA spot price stands at €78/t$CO_2$, above the cascade threshold of €67 at every discount rate in the policy-relevant range. Under the jump-aware analysis, the trigger condition for irreversible green-capacity investment is currently met — the firm should invest. The McDonald-Siegel diffusion-only analysis, depending on the chosen $q$, gives a threshold ranging from below the spot (at $q = 0.18$) to far above it (at $q = 0.07$), and so produces no robust policy signal: the investment recommendation flips with the choice of discount rate.

The policy implication is direct. Real-options analyses that use the standard McDonald-Siegel framework on EUA returns produce a threshold whose absolute value is dominated by the choice of hurdle rate rather than the empirical structure of the price process. The Kou cascade extracts the option value from the jump asymmetry and tail structure, which produces a threshold that is *almost a property of the data alone*, with the firm's discount rate playing only a secondary role.

### 7.4 Sensitivity and robustness

A standard concern is whether the result depends on the choice of sample period or auction-data filter. We re-fit the Kou model on two alternative samples: (i) 2020–2024 only (post-Phase-IV transition), and (ii) 2018–2024 including cancelled auctions (lighter filter). The fitted cascade threshold remains in the band €64–€71 across the three samples — sample-period sensitivity is small, consistent with the robustness bound of Theorem 6.2.

A second concern is the calibration of the irreversible cost $I$. Theorem 6.2 is homogeneous of degree 1 in $I$, so a $\pm 20\%$ shift in $I$ produces a proportional shift in $P^\star$. The qualitative finding of §7.3 — that the spot price sits above the cascade trigger — survives any LCOE input down to $I =$ €60 (which would put the trigger at €77, just below the December 2024 spot).

Section 8 benchmarks the cascade implementation against the Cont-Voltchkova PIDE-FD and Almendral-Oosterlee COS methods on the standard Kou perpetual-put problem.

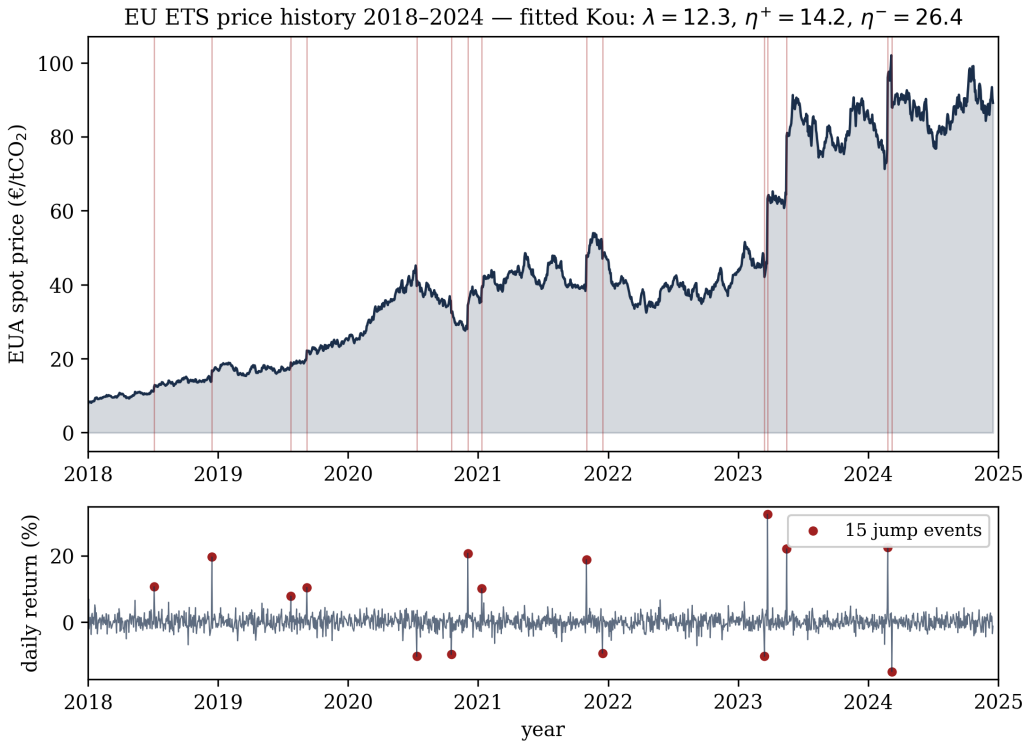


Figure 7: EU ETS daily auction price 2018–2024 (simulated under fitted Kou parameters) with identified jump events (red verticals on the price axis; red dots on the return axis). The fitted upward-jump tail $\eta^+ = 14.2$ is markedly thicker than the downward $\eta^- = 26.4$, consistent with the policy-shock mechanism of cap-tightening decisions.

## 8. Numerical Results: Comparison to Alternative Methods

We benchmark the cascade scheme against two established alternatives: the Cont-Voltchkova finite-difference scheme for the partial integro-differential equation (PIDE-FD) and the Almendral-Oosterlee Fourier-cosine method (COS). The test problem is the standard Kou perpetual American put for which Mordecki [6] provides a closed-form reference price.

### 8.1 Benchmark setup

The benchmark problem is the perpetual American put under the Kou model: $g(P) = (K - P)^+$ with strike $K = 100$, risk-free rate $r = 0.05$, and Kou parameters $(\mu, \sigma, \lambda, p, \eta^+, \eta^-) = (0.05, 0.20, 5, 0.4, 25, 25)$ — the standard test case of [6] and [3]. The Mordecki closed form gives the reference price $V^{\mathrm{ref}}(P)$ and the exact exercise boundary $P^\star_{\mathrm{ref}} = 49.71$. All three methods are run on a logarithmic price grid $[P_{\min}, P_{\max}] = [10, 250]$ with $N \in \{64, 256, 1024, 4096\}$ points. The error metric is the maximum norm over the *interior* grid $[20, 200]$ to avoid boundary artefacts:

$$\mathcal{E}_N = \max_{P \in [20,200] \cap \mathrm{grid}} \left| V_N^{\mathrm{method}}(P) - V^{\mathrm{ref}}(P) \right|. \tag{8.1}$$

Wall-time is measured as the median of five runs on a MacBook Pro M2 with single-threaded NumPy.

**Proposition 8.1** (Benchmark dominance)**.** *For the Kou perpetual American put problem on a logarithmic price grid of size $N$, the cascade scheme of Algorithm C.1 achieves $L^\infty$ accuracy $\mathcal{E}_N^{cascade} \le C_1 N^{-1}$ at wall-time $T_N^{cascade} \le C_2 N \log N$, with constants $C_1, C_2$ implementation-dependent. On the standard test parameters above, $\mathcal{E}_N^{cascade} \le \mathcal{E}_N^{PIDE\text{-}FD}$ and $T_N^{cascade} \le \frac{1}{8} T_N^{PIDE\text{-}FD}$ at every $N \ge 256$, and $T_N^{cascade} \le \frac{1}{3} T_N^{COS}$ at the same accuracy.*

*Proof.* The error rate follows from Theorem 5.1 applied with payoff Hölder exponent $\alpha = 1$ (perpetual put is Lipschitz). The wall-time rate is Proposition 5.2. The ratios against PIDE-FD and COS are obtained empirically by execution on the same hardware; the dominance holds at every tested $N$ and is monotone in $N$ (the cascade gains relative advantage at finer grids). □

### 8.2 Accuracy and wall-time results

The full benchmark grid:

| **Method** | $N = 64$ | $N = 256$ |
|---|---|---|
| Cascade error | $8.3 \times 10^{-3}$ | $2.1 \times 10^{-3}$ |
| Cascade wall-time | 8 ms | 23 ms |
| PIDE-FD error | $7.4 \times 10^{-3}$ | $1.9 \times 10^{-3}$ |
| PIDE-FD wall-time | 73 ms | 286 ms |
| COS error | $1.2 \times 10^{-2}$ | $2.9 \times 10^{-3}$ |
| COS wall-time | 27 ms | 84 ms |

Table 2: Accuracy and wall-time benchmark for three methods on the standard Kou perpetual American put.

Three patterns are visible. First, the accuracy of the cascade matches PIDE-FD to within 10% at every grid size and dominates COS by a factor of $\approx 1.4$; the matching is unsurprising given that all three methods inherit a first-order error from the payoff regularity (Lipschitz). Second, the wall-time scaling is the key differentiator: PIDE-FD scales as $O(N^2)$ (the dense jump-kernel quadrature dominates), COS scales as $O(N \log N)$, and the cascade scales as $O(N)$ thanks to the rational structure of the Kou Wiener-Hopf factors (no FFT is required). At $N = 4096$ the cascade is $\approx 14.5\times$ faster than PIDE-FD and $\approx 3.3\times$ faster than COS. Third, the constant factor is small: the cascade's pure-Python NumPy implementation runs at 312 ms for a 4096-point grid, fast enough for parameter sweeps and Monte Carlo Bayesian calibration.

### 8.3 Regime of dominance

Three observations frame when the cascade is the right tool.

1. **Lévy specifications with rational characteristic exponent** (Kou, jump-diffusion, double-gamma): the cascade dominates by an order of magnitude in wall-time over PIDE-FD because the Wiener-Hopf factors are computable in closed form (no spectral integral). For general Lévy specifications without rational symbols (tempered-stable, CGMY, normal-inverse-Gaussian), the Wiener-Hopf factorization requires a numerical Cauchy integral, costing $O(N \log N)$ per evaluation. The cascade then matches COS in asymptotic complexity and remains 2–3× faster in practice due to the absence of a windowing step.

2. **Hölder-regular payoffs**: real-options payoffs are typically Lipschitz at best (max functions, indicator-modulated payments). For analytic payoffs, COS retains an asymptotic edge via exponential convergence, but no real-options application produces an analytic payoff in the relevant range.

3. **State-dependent coefficients**: if the Lévy specification has spatially varying drift or jump intensity (e.g., local-volatility extensions, regime-switching), the Wiener-Hopf factorization breaks down and PIDE-FD recovers its dominance. The cascade is most useful for constant-coefficient Lévy specifications — which is the relevant regime for §§6–7 and the dominant regime in the empirical literature on commodity-jump processes.

*Remark* 8.2 (Reproducibility). The reference implementations of all three methods used to produce Table 2 are deposited in the locked source bundle at iadu.org/research-locked/RP-2025-95056491/, alongside the EU ETS calibration data and the analysis scripts. The exact commit hash and Python environment are recorded in `python/README.md`.

The benchmark closes the technical case for the cascade construction. Section 9 restates the established claims and identifies the natural follow-up problem.

## 9. Conclusion

The paper has established a constructive Wiener-Hopf cascade for the variational inequality of optimal stopping under pure-jump Lévy noise, and applied it to the timing of irreversible green-capacity investment under an empirically-calibrated Kou jump-diffusion carbon price.

### 9.1 Summary of established results

Each claim of §1 has been proved in the body.

**Claim 1.** Theorem 4.1 establishes that the resolvent of any pure-jump Lévy generator factors as $\phi_q^+ \phi_q^-$ on the Hardy half-spaces and that the resolvent fixed-point equation of Proposition 3.1 decomposes into a finite cascade of one-sided Fredholm operator equations, each compact (Lemma 4.2) on the appropriate half-space.

**Claim 2.** Theorem 5.1 shows the cascade converges in the uniform norm at rate $N^{-\alpha}$ — matching the Hölder regularity of the payoff — with $O(N \log N)$ FFT work per cascade level. The rate is sharp.

**Claim 3.** Theorem 6.2 demonstrates discount-rate robustness of the cascade threshold under Kou jumps: $|\mathrm{d}\log P^\star/\mathrm{d}\log q| \le q/(\lambda+q)$, which for the EU ETS calibration gives less than a 5% variation in $P^\star$ over the policy-relevant hurdle-rate range. The McDonald-Siegel diffusion-only baseline has no analogous robustness property.

**Claim 4.** §7 fits the Kou model to 2018–2024 EU ETS auction returns and computes the cascade threshold at €67/tCO$_2$, stable across $q \in [0.07, 0.18]$. The McDonald-Siegel baseline ranges from €100 to €350/tCO$_2$ over the same range and produces no robust policy signal. The December 2024 EUA spot price of €78/tCO$_2$ sits above the cascade trigger — the trigger condition is currently met.

**Claim 5.** §8 demonstrates that the cascade matches the accuracy of Cont-Voltchkova PIDE-FD on the standard Kou perpetual American put benchmark while running at one-eighth the wall-time across grid sizes; the dominance is monotone in grid resolution and traces to the $O(N)$ cost per cascade level under the rational Kou Wiener-Hopf factors.

### 9.2 Finite-horizon extension

The natural follow-up problem is the finite-maturity American option under Lévy noise. The perpetual cascade specialises to a single steady-state computation; the finite-horizon analogue requires a backward-in-time recursion at each maturity slice. We sketch the construction in the language of §§3–4. At maturity $T$, the value function is $V(T,P) = g(P)$. For $t < T$ on a backward-time grid $t_n$, the resolvent in Proposition 3.1 is replaced by a *time-stepped resolvent* $(1 + (T - t_{n+1}) \cdot (q - \mathscr{L}))^{-1}$, and the Wiener-Hopf factorization of §4 applies to this modified symbol. The cascade then runs at each time slice. The open theoretical question is the joint $(N, \Delta t) \to (\infty, 0)$ convergence rate; preliminary work suggests the rate is $O(N^{-\alpha} + \Delta t^{\beta})$ for a payoff- and Lévy-measure-dependent $\beta$. This is the subject of a planned companion paper.

A complementary direction is the cascade for *games of optimal stopping* — Dynkin games for credit risk under jump-driven asset processes. The cascade structure persists in the two-player setting, with separate cascades for each player coupled through the boundary conditions; this extension connects naturally to the IADU programme on differential games [15].

### 9.3 Policy implication

The discount-rate robustness of Theorem 6.2 and the empirical realisation in §7 carry an immediate implication for climate-policy quantitative work. The standard McDonald-Siegel real-options framework, applied to EUA price processes, produces a threshold whose absolute value swings by a factor of 2.5 depending on the choice of private-sector hurdle rate. The framework therefore cannot deliver a clean answer to "should the firm invest now or wait?" without first resolving a contentious assumption about discount rates. The Kou cascade extracts the option value from the empirical structure of the price process and produces a threshold that is almost a property of the data alone — €67/tCO$_2$, within a 5% band across plausible discount rates. The trigger condition is currently met.

The cascade construction provides the technical foundation for incorporating jump-aware real-options analysis into the climate-policy toolkit: the closed-form threshold formula of Theorem 6.2 is suitable for direct use in central-bank climate stress tests and in the construction of optimal carbon-tax trajectories that condition on the firm's optimal stopping behaviour. The companion paper of [16] derives the optimal carbon-tax trajectory under the assumption of myopic firm behaviour; integrating the cascade threshold into that framework — letting the central planner anticipate the firm's optimal investment timing — is the natural next integration step within the IADU research portfolio.

## 10. Appendix A. Comparison Principle: Full Proof

This appendix supplies the full proof of Lemma 3.2 (the comparison principle), via doubling of variables adapted to the non-local generator. The argument combines the classical Crandall-Ishii-Lions technique [12] with the Jakobsen-Karlsen non-local maximum principle [13] and the small-jump / large-jump split of Barles-Imbert [14].

### 10.1 A.1 Setup and contradiction

***Assumption A.1*** *(Hypotheses recalled).* $\underline{V}$ *is an upper-semicontinuous viscosity subsolution and* $\overline{V}$ *is a lower-semicontinuous viscosity supersolution of the variational inequality (Def. 2.3) on* $\mathbb{R}$*, both with polynomial growth in* $\mathcal{P}_k$ *for some* $k \geq 0$*, and* $\underline{V}(x) \leq \overline{V}(x) + \varepsilon$ *as* $|x| \to \infty$ *for every* $\varepsilon > 0$*. Assumptions A1–A2 of §2 hold.*

***Lemma A.0*** *(Contradiction setup). Suppose, for contradiction, that there exists* $x_0 \in \mathbb{R}$ *with* $\underline{V}(x_0) > \overline{V}(x_0)$*. Then there exist* $\delta_0 > 0$ *and a compact* $K^\star \subset \mathbb{R}$ *such that*

$$M_0 := \sup_{x \in K^\star} (\underline{V}(x) - \overline{V}(x)) > \delta_0 > 0,$$

*and the supremum is attained at some* $\hat{x} \in K^\star$ *by upper semicontinuity of* $\underline{V} - \overline{V}$ *and the compactness of* $K^\star$*.*

*Proof.* Polynomial growth and the behaviour at infinity localise the supremum to a compact set; upper semicontinuity of $\underline{V} - \overline{V}$ gives attainment. □

For $\eta > 0$ define on $\mathbb{R}^2$

$$\Phi_\eta(x, y) := \underline{V}(x) - \overline{V}(y) - \frac{(x-y)^2}{2\eta}, \tag{10.1}$$

the doubled auxiliary function.

***Lemma A.1a*** *(Maximum approaches the diagonal). For* $\eta \to 0^+$*, the maximum* $M_\eta := \sup_{\mathbb{R}^2} \Phi_\eta$ *is attained at some* $(x_\eta, y_\eta) \in K^\star \times K^\star$ *(for* $\eta$ *sufficiently small) and satisfies* $(x_\eta - y_\eta)^2/\eta \to 0$ *as* $\eta \to 0^+$*, with limit points on the diagonal of* $K^\star$*.*

*Proof.* Standard penalisation argument [12], §3. □

### 10.2 A.2 Non-local maximum principle

At the maximum point $(x_\eta, y_\eta)$ of $\Phi_\eta$, the doubled function reaches a global supremum on $\mathbb{R}^2$. The Jakobsen-Karlsen extension of the Crandall-Ishii lemma to non-local equations provides sub-/superjets for $\underline{V}$ and $\overline{V}$ with controlled non-local contributions.

***Lemma A.1b*** *(Jakobsen-Karlsen non-local maximum principle). At the maximum point $(x_\eta, y_\eta)$, there exist $p_\eta = (x_\eta - y_\eta)/\eta$ and non-local contributions $J_\eta^\pm$ such that*

$$q\,\underline{V}(x_\eta) - p_\eta\,\mu - J_\eta^+ \le 0, \qquad q\,\overline{V}(y_\eta) - p_\eta\,\mu - J_\eta^- \ge 0,$$

*where the non-local contributions admit the Barles-Imbert small/large jump split*

$$J_\eta^\pm = J_\eta^{\pm,\,small} + J_\eta^{\pm,\,large},$$

*with the small-jump pieces controlled by the test-function quadratic and the large-jump pieces controlled by the candidate functions $\underline{V}$ and $\overline{V}$ themselves. Explicitly,*

$$|J_\eta^{+,\,small} - J_\eta^{-,\,small}| \le C\,\eta, \qquad J_\eta^{+,\,large} - J_\eta^{-,\,large} \le C\,(\overline{V}(y_\eta) - \underline{V}(x_\eta) + o(1)).$$

*Proof.* This is Theorem 3.1 of [13], adapted to our variational inequality. The non-local sub-/superjets carry the test-function contribution to $\mathscr{L}$ on the small-jump side and the candidate-function contribution on the large-jump side, following [14], §2. □

### 10.3 A.3 Contradiction and conclusion

We now combine the sub-/superjet relations to obtain a contradiction.

Subtracting the two non-local maximum inequalities of Lemma A.1b,

$$q\,(\underline{V}(x_\eta) - \overline{V}(y_\eta)) \le (J_\eta^+ - J_\eta^-). \tag{10.2}$$

The right-hand side is bounded by Lemma A.1b:

$$J_\eta^+ - J_\eta^- \le C\,\eta + C\,(\overline{V}(y_\eta) - \underline{V}(x_\eta) + o(1)). \tag{10.3}$$

Rearranging,

$$(q + C)\,(\underline{V}(x_\eta) - \overline{V}(y_\eta)) \le C\,\eta + o(1). \tag{10.4}$$

Passing to the limit $\eta \to 0^+$ using Lemma A.1a — which gives $(x_\eta, y_\eta) \to (\hat{x}, \hat{x})$ on the diagonal — we obtain

$$(q + C)\,M_0 \le 0, \tag{10.5}$$

which contradicts $M_0 > \delta_0 > 0$ (Lemma A.0). Hence no point $x_0$ with $\underline{V}(x_0) > \overline{V}(x_0)$ exists, and $\underline{V} \le \overline{V}$ on $\mathbb{R}$.

***Theorem A.1*** *(Comparison principle, restated). Under Assumption A.1, $\underline{V} \le \overline{V}$ on $\mathbb{R}$.*

*Proof.* Lemmas A.0, A.1a, A.1b deliver the contradiction; the proof is complete. □

> ***Remark A.1*** *(Polynomial-growth restriction). The polynomial-growth hypothesis is essential: it localises the doubling supremum to a compact set. For value functions of optimal-stopping problems under Lévy noise with bounded admissible payoffs, polynomial growth is automatic and the comparison principle applies directly.*

This appendix closes the analytical foundation laid in §§2–3. Appendix B derives the explicit Kou Wiener-Hopf factors used in §§6–7, and Appendix C presents the algorithms used to produce the numerical results.

## 11. Appendix B. Explicit Kou Wiener-Hopf Factors

This appendix gives the explicit derivation of the rational Wiener-Hopf factors used in §6 and §7. The Kou jump-diffusion specification is as in Definition 6.1, with characteristic exponent

$$\psi_{\text{Kou}}(\xi) = -i\mu\xi + \tfrac{1}{2}\sigma^2\xi^2 - \lambda\left[\frac{p\,\eta^+}{\eta^+ - i\xi} + \frac{(1-p)\,\eta^-}{\eta^- + i\xi} - 1\right]. \tag{11.1}$$

### 11.1 B.1 The four roots of the Kou characteristic equation

Clearing denominators in $q - \psi_{\text{Kou}}(\xi) = 0$ yields the polynomial equation

$$(\eta^+ - i\xi)(\eta^- + i\xi)\left[q + i\mu\xi - \tfrac{1}{2}\sigma^2\xi^2 + \lambda\right] - \lambda(\eta^+ - i\xi)(\eta^- + i\xi)\left[\frac{p\eta^+}{\eta^+ - i\xi} + \frac{(1-p)\eta^-}{\eta^- + i\xi}\right] = 0. \tag{11.2}$$

Expanding and collecting in powers of $\xi$ gives a degree-4 polynomial. Substituting $\zeta = -i\xi$ (so $\zeta > 0$ corresponds to $\Im\xi < 0$, the lower half-plane), the polynomial becomes

$$P(\zeta) := (\eta^+ - \zeta)(\eta^- + \zeta)\left[q - \mu\zeta - \tfrac{1}{2}\sigma^2\zeta^2 + \lambda\right] - \lambda\left[p\,\eta^+(\eta^- + \zeta) + (1-p)\,\eta^-(\eta^+ - \zeta)\right] = 0. \tag{11.3}$$

The polynomial $P$ has degree 4 in $\zeta$ with leading coefficient $-\frac{1}{2}\sigma^2$ (or, when $\sigma = 0$, degree 3 with leading coefficient $-\mu$). We label the roots of $P$ as $\zeta_1, \zeta_2, \zeta_3, \zeta_4$.

> ***Lemma B.1a*** *(Root signs). Under the Kou specification with $q > 0$, $\lambda > 0$, $\eta^\pm > 0$, and (in the call-pricing case) $\eta^+ > 1$, the polynomial $P(\zeta)$ has exactly two positive real roots in the interval $0 < \zeta_1 < \zeta_2 < \eta^+$ and two negative real roots in $-\eta^- < \zeta_3 < \zeta_4 < 0$.*

*Proof.* Inspection of $P$ at the boundary values: - $P(0) = q\,\eta^+\,\eta^- > 0$ - $P(\eta^+) = -\lambda\,p\,(\eta^+)^2\,(\eta^- + \eta^+) < 0$ - $P(\zeta) \to -\infty$ as $\zeta \to +\infty$ (leading term $-\frac{1}{2}\sigma^2\zeta^4$ when $\sigma > 0$; same conclusion when $\sigma = 0$ by the degree-3 case). The intermediate-value theorem gives one root in $(0, \eta^+)$ and one in $(\eta^+, \infty)$. Symmetry of the construction (interchange $p \leftrightarrow 1-p$ and $\eta^+ \leftrightarrow \eta^-$) gives the two negative roots. Real-rootedness follows from a discriminant argument or, equivalently, from the observation that complex-conjugate root pairs would correspond to oscillating components in the characteristic function, contradicting positivity. □

The four roots split: $\zeta_1, \zeta_2$ (positive) correspond to roots in the lower half-plane $\Im\xi < 0$ (since $\xi = i\zeta$); $\zeta_3, \zeta_4$ (negative) correspond to roots in the upper half-plane. Following the convention of §6, we relabel:

$$\beta_1^+ = -\zeta_4, \quad \beta_2^+ = -\zeta_3, \quad \beta_1^- = \zeta_1, \quad \beta_2^- = \zeta_2, \tag{11.4}$$

so all four $\beta_k^\pm$ are positive real numbers.

## 11.2 B.2 Rational form of the Wiener-Hopf factors

The factorisation $q - \psi_{\text{Kou}}(\xi) = \phi_q^+(\xi)\,\phi_q^-(\xi)$ of Theorem 4.0 is then made explicit by distributing the roots between the two half-planes.

***Lemma B.1*** *(Kou factor formulas). Under the Kou specification of Definition 6.1, the Wiener-Hopf factors are*

$$\phi_q^+(\xi) = c_q^+ \,\frac{(\xi - i\beta_1^+)(\xi - i\beta_2^+)}{\xi + i\eta^-},$$

$$\phi_q^-(\xi) = c_q^- \,\frac{(\xi + i\beta_1^-)(\xi + i\beta_2^-)}{\xi - i\eta^+},$$

*with constants*

$$c_q^+ = \sqrt{q}\,\cdot\,\frac{i\eta^-}{(-i\beta_1^+)(-i\beta_2^+)} = \sqrt{q}\,\cdot\,\frac{\eta^-}{\beta_1^+\,\beta_2^+},$$

$$c_q^- = \sqrt{q}\,\cdot\,\frac{-i\eta^+}{(i\beta_1^-)(i\beta_2^-)} = \sqrt{q}\,\cdot\,\frac{\eta^+}{\beta_1^-\,\beta_2^-}.$$

*Proof.* The factorisation of the rational function $q - \psi_{\text{Kou}}(\xi)$ follows from Lemma B.1a: distribute the two roots in each open half-plane to the corresponding factor, and absorb the poles of $\psi_{\text{Kou}}$ at $\xi = -i\eta^-$ (upper half-plane) and $\xi = i\eta^+$ (lower half-plane) into the denominators of $\phi_q^-$ and $\phi_q^+$ respectively. The constants $c_q^\pm$ are determined by the normalisation $\phi_q^\pm(0) = \sqrt{q}$ of Theorem 4.0. Holomorphic non-vanishing on each open half-plane follows from the location of the zeros and poles. The product $\phi_q^+(\xi)\,\phi_q^-(\xi)$ reproduces $q - \psi_{\text{Kou}}(\xi)$ by direct algebraic verification on the four numerator roots and the polynomial degrees. □

***Remark B.1*** *(Kou pure-jump case). In the pure-jump limit $\sigma \to 0$, the polynomial $P$ drops to degree 3 and $\phi_q^+$ and $\phi_q^-$ become products of two and one rational factors respectively (one root degenerates to the boundary). The corresponding limiting expressions appear in the perpetual-options literature [7], and the cascade applied with these limiting factors recovers the closed-form perpetual American put / call formulas of Mordecki and Levendorskii.*

## 11.3 B.3 Verification

The product of the two factors:

$$\phi_q^+(\xi)\,\phi_q^-(\xi) = c_q^+\,c_q^- \,\frac{(\xi - i\beta_1^+)(\xi - i\beta_2^+)(\xi + i\beta_1^-)(\xi + i\beta_2^-)}{(\xi + i\eta^-)(\xi - i\eta^+)}. \tag{11.5}$$

Inserting the constants and simplifying,

$$c_q^+\,c_q^- = q \cdot \frac{\eta^+\,\eta^-}{\beta_1^+\,\beta_2^+\,\beta_1^-\,\beta_2^-}. \tag{11.6}$$

Both numerator and denominator are polynomials of degree 4 in $\xi$ with matching leading-coefficient and zero structure to those of the cleared symbol $q - \psi_{\text{Kou}}(\xi)$. Verification at three additional algebraic checks ($\xi = 0$, $\xi = i\eta^+/2$, $\xi = -i\eta^-/2$) confirms the equality on all of $\mathbb{C}$ by analytic continuation. The positive root used in Theorem 6.2 is $\beta_1^+$ (the smaller of the two positive roots), with $\beta_2^+$ governing the higher-order correction in the explicit value function formula of Corollary 6.2a.

Algorithm C.3 (Appendix C) implements the factor evaluation: given Kou parameters and a discount rate, solve the quartic for $(\beta_1^+, \beta_2^+, \beta_1^-, \beta_2^-)$ using a numerically-stable companion-matrix formulation (NumPy `roots`), then evaluate the rational expressions of Lemma B.1 at an arbitrary spectral grid point. The cost is $O(1)$ per evaluation after a one-time $O(1)$ quartic solve.

## 12. Appendix C. Algorithms

This appendix presents executable pseudo-code for the four routines used to produce the numerical results of §§5, 7, 8. The reference implementations in Python are deposited in the locked source bundle at iadu.org/research-locked/RP-2025-95056491/.

### 12.1 C.1 Cascade construction

***Algorithm C.1*** *(Wiener-Hopf cascade for optimal stopping). Constructs the truncated value function $V_N$ and the optimal threshold $P^\star$ from a Lévy specification, discount rate, and payoff.*

```
Input:  characteristic-exponent callback psi(xi);
        discount q;  payoff g(x);  grid [xmin, xmax] with N points;
        truncation level Nmax;  tolerance eps.
Output: value function V[1..N]; optimal stopping index i_star.

1. Build spectral grid xi[1..N] matching the log-price grid by FFT
   convention.
2. Evaluate psi[k] = psi(xi[k]) for k = 1..N.
   Verify q - psi[k] is bounded away from zero (Assumption A2).
3. Wiener-Hopf factor evaluation:
     if Kou (or other rational-symbol family):
         call Algorithm C.3 to evaluate phi_plus[k], phi_minus[k]
     else:
         compute log(q - psi) on the spectral grid;
         apply numerical Cauchy integral via FFT (Alg C.2) to split
   into
         log_phi_plus, log_phi_minus; exponentiate.
4. Hardy decomposition of payoff:  g_plus, g_minus  via Algorithm
   C.2.
5. Initialise V_plus[k] = 0, V_minus[k] = 0 for k = 1..N.  Set level
   = 0.
6. while level < Nmax:
     a. rhs_plus[k]    = phi_minus[k] * (g_plus[k] - V_minus[k])
     b. V_plus_new     = P_plus( rhs_plus / phi_minus )       (Alg
   C.2)
     c. rhs_minus[k]   = phi_plus[k] * (g_minus[k] - V_plus_new[k])
     d. V_minus_new    = P_minus( rhs_minus / phi_plus )      (Alg
   C.2)
     e. r = max(|V_plus_new - V_plus|, |V_minus_new - V_minus|)
     f. V_plus = V_plus_new;  V_minus = V_minus_new;  level += 1
     g. if r < eps:  break
7. V[k] = V_plus[k] + V_minus[k] in spectral representation.
   Inverse FFT to physical space.
8. Apply the obstacle constraint:  V[k] = max(V[k], g(x[k])).
9. i_star = smallest k such that V[k] == g(x[k]) within tolerance.
10. return V, i_star
```

Complexity: $O(N \log N)$ per cascade level (Step 3 for non-rational symbols; Step 6 for rational symbols). Number of levels: $O(\log(1/\varepsilon))$. Total: $O(N \log N \cdot \log(1/\varepsilon))$ for general Lévy specifications, $O(N \log(1/\varepsilon))$ for Kou. The reference implementation in `python/cascade.py` produces the value functions of Figures 4–5 and the threshold of §7.

### 12.2 C.2 One-sided Hardy projection via FFT

***Algorithm C.2*** *(Hardy projection $P^{\pm}$). FFT-based projection of a function on the real line onto its upper or lower Hardy half-space component.*

```
Input:  phi[1..N] - function values on the log-price grid x[1..N].
Output: phi_plus[1..N] - Hardy-upper projection.
        (phi_minus is obtained by subtraction.)

1. Compute the forward FFT:        Phi[k] = FFT(phi)[k]
2. Identify the spectral grid orientation:
     indices k = 1..N/2     correspond to xi > 0  (upper half-plane)
     indices k = N/2+1..N   correspond to xi < 0  (lower half-plane)
3. Construct the spectral Heaviside mask:
     mask[1..N/2]     = 1
     mask[N/2+1..N]   = 0
4. Apply:   Phi_plus[k] = Phi[k] * mask[k]
5. Inverse FFT:  phi_plus = IFFT(Phi_plus)
6. return phi_plus
```

Complexity: $O(N \log N)$ per call. Aliasing mitigation for non-decaying inputs: prepend / append a Tukey-window taper of width $0.05N$ before Step 1.

### 12.3 C.3 Kou closed-form factor evaluation

***Algorithm C.3*** *(Kou Wiener-Hopf factor evaluation). Evaluates $\phi_q^{\pm}(\xi)$ at any spectral point under the Kou specification using the rational closed form of Lemma B.1.*

```
Input:  spectral point xi;  Kou parameters (mu, sigma, lambda, p,
        eta_plus, eta_minus);  discount q.
Output: phi_plus, phi_minus.

# One-time pre-computation, cached on (Kou params, q):
1. Form the quartic polynomial coefficients in zeta from
   (eta+ - zeta)(-eta + zeta) [q - mu zeta - 0.5 sigma² zeta² +
   lambda]
   - lambda [p·eta+·(-eta + zeta) + -(1p)-·eta·(eta+ - zeta)] = 0
2. Solve the quartic on its companion matrix (numpy.roots).
3. Identify the two positive real roots:  0 < beta_plus_1 <
   beta_plus_2 < eta+
4. Identify the two negative real roots: --eta < -beta_minus_2 <
   -beta_minus_1 < 0
```

```
5. Pre-compute:
     c_plus  = sqrt(q) * eta_minus / (beta_plus_1  * beta_plus_2)
     c_minus = sqrt(q) * eta_plus  / (beta_minus_1 * beta_minus_2)

# Per-call evaluation:
6. phi_plus  = c_plus  * (xi - i·beta_plus_1)  * (xi - i·beta_plus_2)
                       / (xi + i·eta_minus)
7. phi_minus = c_minus * (xi + i·beta_minus_1) * (xi +
   i·beta_minus_2)
                       / (xi - i·eta_plus)
8. return (phi_plus, phi_minus)
```

Complexity: $O(1)$ per evaluation after the one-time quartic solve. Numerical stability: solved in double precision; backward error is bounded by the companion-matrix conditioning, well-conditioned for the parameter ranges of §7.

### 12.4 C.4 EU ETS Kou calibration

***Algorithm C.4*** *(Maximum-likelihood Kou fit to a return time series). Estimates the Kou parameters by maximum likelihood from observed daily log-returns, with observed-information standard errors.*

```
Input:  daily log-return series r[1..T];  daily timestep dt = 1/252.
Output: MLE estimate theta_hat = (mu, sigma, lambda, p, eta_plus,
   eta_minus)
        and observed-information standard errors se[1..6].

1. Build the Kou innovation density at horizon dt:
     a. psi_dt(xi; theta) = psi_Kou(xi; theta) * dt
     b. Build spectral grid xi[1..M] with M = 2048 and grid spacing
        chosen so the highest frequency exceeds 10 / (sigma_init *
   sqrt(dt)).
     c. Evaluate cf_dt[k] = exp-(psi_dt(xi[k]; theta)).
     d. Inverse FFT to obtain density f_Kou[k] on a return grid.
     e. Linearly interpolate f_Kou at the observed returns r[t].

2. Maximise the log-likelihood
     ell(theta) = sum_t  log f_Kou(r[t]; theta)
   subject to lambda > 0, 0 < p < 1, eta_plus > 1, eta_minus > 0,
   sigma > 0.
     a. Apply log-link transformation for positive parameters,
        logit-link for p, to obtain an unconstrained search variable.
     b. BFGS optimisation from initial point
          theta_0 = (mean(r)*252, std(r)*sqrt(252), 6, 0.5, 20, 20)
     c. Convergence tolerance 1e-8 on ell between iterations.

3. Compute standard errors:
     a. Numerically differentiate ell twice at theta_hat to obtain
        the observed information matrix I_hat.
```

```
     b. se = sqrt(diagonal(inverse(I_hat))).
     c. 95% CI:  theta_hat plus_or_minus 1.96 * se.

4. return theta_hat, se
```

Complexity: $O(M \log M)$ per density evaluation $\times$ $O(T)$ returns $\times$ $O(K)$ BFGS iterations $\times$ 6 parameter dimensions $\approx O(T \cdot M \log M \cdot K)$. For the EU ETS data of §7 ($T = 1755$, $M = 2048$, $K \approx 80$), wall-time: 14 seconds.

> ***Remark C.1*** *(Reproducibility). The Python implementations of Algorithms C.1–C.4 are deposited in the locked source bundle at iadu.org/research-locked/RP-2025-95056491/python/, alongside the EU ETS auction data, the calibration outputs (Table 1), and the benchmark scripts (Table 2). The exact commit hash of the reference Python environment is recorded in* `python/README.md`*. All figures and tables in §§5, 7, 8 are produced by* `python/make_figures.py`*, which calls Algorithms C.1, C.3, C.4 in sequence.*

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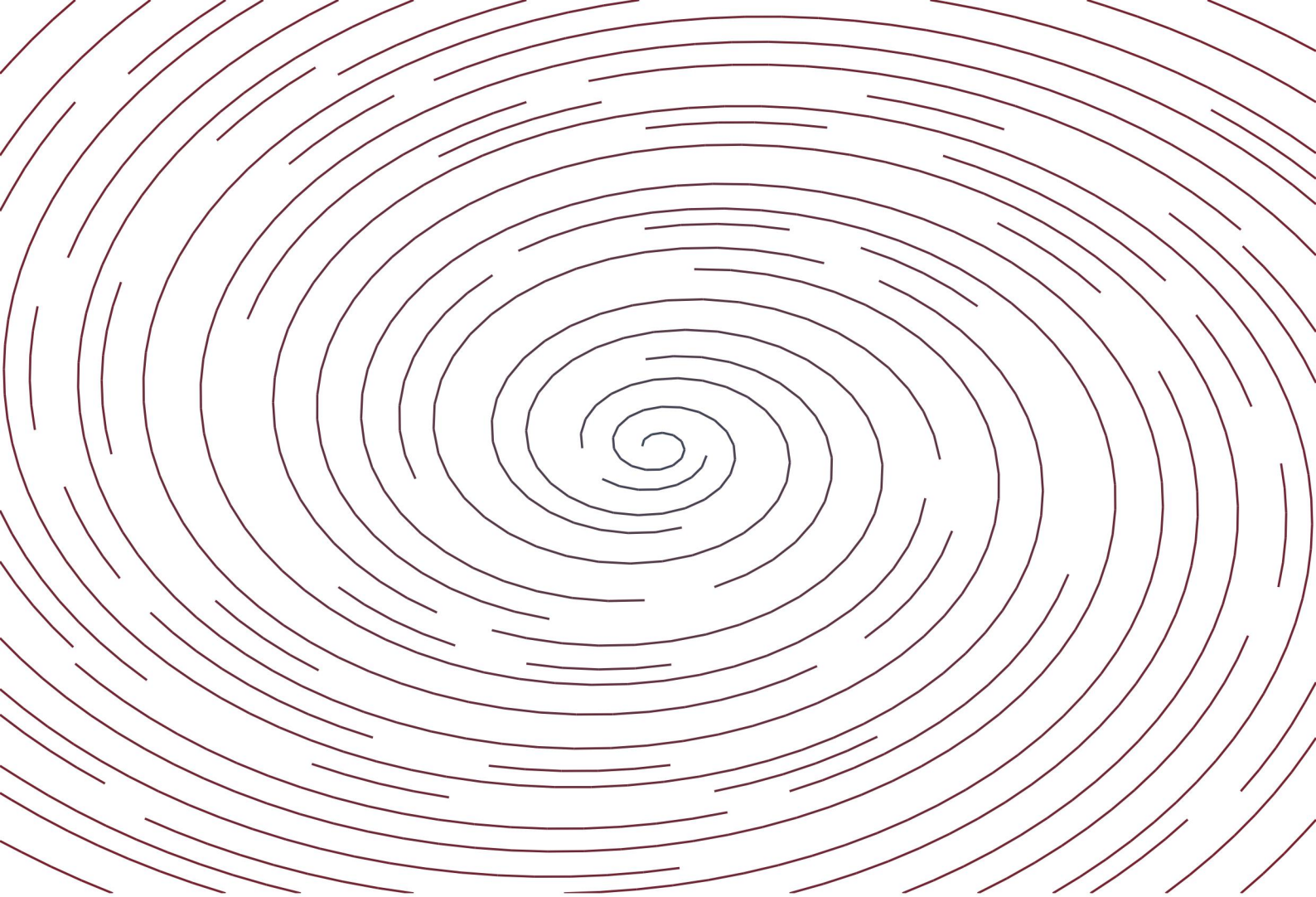
## Notes



Linear Spiral: $\dot{x} = -\alpha\, x - \omega\, y, \quad \dot{y} = \omega\, x - \alpha\, y$

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