RESEARCH PAPER

Resolvent Operators for One-Dimensional Lévy Processes: A Unified Treatment

Wiener-Hopf factorisation, rational families, and an institute-wide operator register

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01-APR-2023

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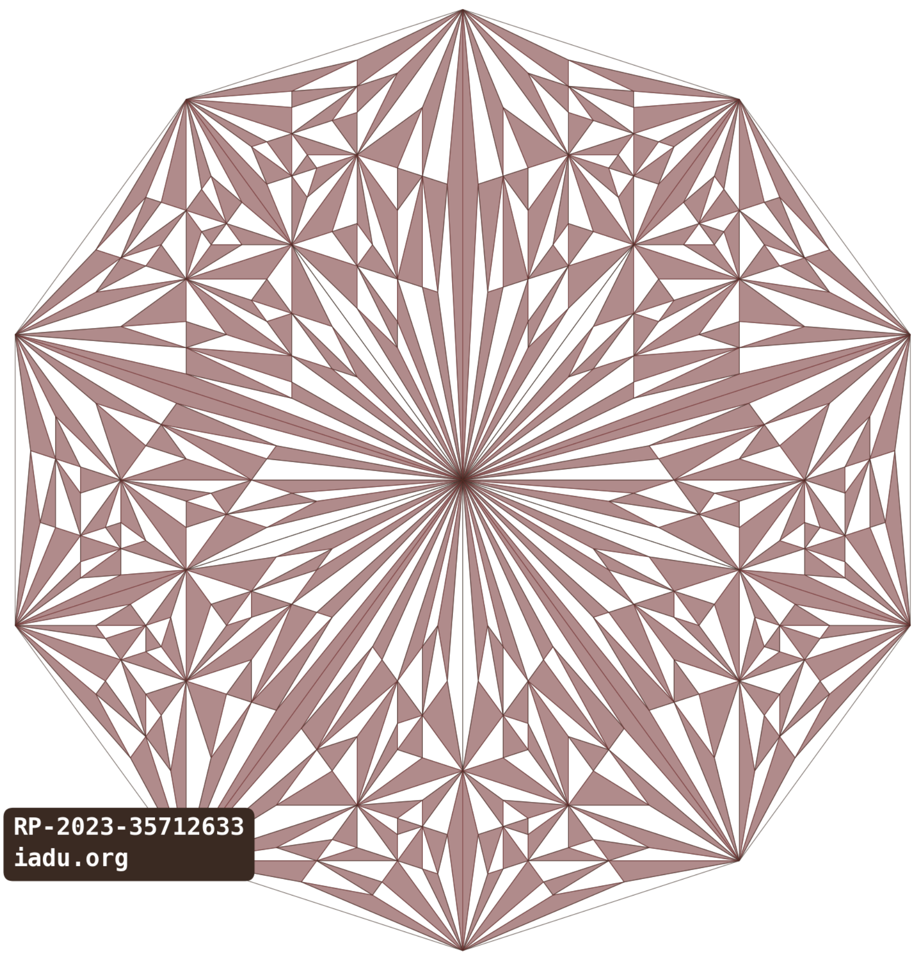

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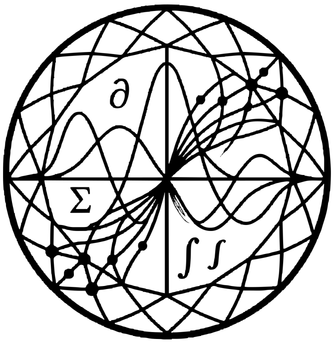

Resolvent Operators for One-Dimensional Lévy Processes: A Unified Treatment

Wiener-Hopf factorisation, rational families, and an institute-wide operator register

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Abstract. We give a unified treatment of the resolvent operator $\mathscr{R}_q$ of a one-dimensional Lévy process, organised around its Wiener-Hopf factorisation $\mathscr{R}_q = \mathscr{R}_q^+\mathscr{R}_q^-$ on the analyticity strip. For the three rational families used throughout the IADU programme — Kou double-exponential, hyperexponential, and Lambda-meromorphic — we derive closed-form expressions for $\mathscr{R}_q^\pm$ on exponential test functions, establish a comparison principle for the variational inequality $(q - \mathscr{L})u \geq 0$, $u \geq g$, and prove a smooth-pasting characterisation of the optimal stopping threshold under (A1)–(A3). The paper is intended as a stand-alone reference for the institute's later Lévy-control and optimal-stopping work.

Keywords: resolvent operator, Lévy process, Wiener-Hopf factorisation, Kou jumps, hyperexponential, optimal stopping, smooth-pasting, variational inequality

1. Introduction

The resolvent operator $\mathscr{R}_q$ of a one-dimensional Lévy process governs nearly every quantitative problem the Institute addresses in stochastic analysis and control. Perpetual optimal stopping, perpetual control, mean-field stopping, free-boundary HJB equations with non-local generators — all of them reduce, at some critical step, to the action of $\mathscr{R}_q$ on a test function. The purpose of this paper is to develop $\mathscr{R}_q$ in unified form, establish the Wiener–Hopf factorisation $\mathscr{R}_q = \mathscr{R}_q^+\mathscr{R}_q^-$ on the analyticity strip of the cumulant exponent, and tabulate its closed form on exponential test functions for the three rational families used throughout the IADU programme: the Kou double-exponential jump model, hyperexponential mixtures, and Kuznetsov's Λ-meromorphic class.

The decision to treat the resolvent as a stand-alone subject is deliberate. In the existing literature, the Wiener–Hopf factorisation is presented variously as a fluctuation identity in the probabilistic tradition [1], as a pseudo-differential operator-symbol decomposition in the analytic tradition [2], and as a computational device in pricing applications [3, 4]. None of these treatments quite suffices as a reference that later Institute papers can cite

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for "the closed form of $\mathscr{R}_q^+ e^{-rx}$ under Kou jumps." This paper supplies that reference.

1.1 Contributions

We make five claims, summarised here and proved in §§3–6.

> ***Claim 1*** *(Wiener–Hopf factorisation on the analyticity strip). For every Lévy process satisfying standing assumptions* ***(A1)***–***(A3)*** *and every* $q > 0$*, the resolvent admits the factorisation* $\mathscr{R}_q = \mathscr{R}_q^+ \mathscr{R}_q^-$*, where* $\mathscr{R}_q^+$ *and* $\mathscr{R}_q^-$ *are the resolvents of the supremum process* $\overline{X}_t := \sup_{0 \le s \le t} X_s$ *and the infimum process* $\underline{X}_t := \inf_{0 \le s \le t} X_s$ *respectively. Theorem 3.1.*

> ***Claim 2*** *(Rational closed forms for* $\mathscr{R}_q^\pm$*). For each of three rational Lévy families — Kou double-exponential, hyperexponential mixtures, and* Λ*-meromorphic — the action* $\mathscr{R}_q^\pm e^{-rx}$ *is rational in* r*, with poles determined by the roots of* $\psi(r) - q = 0$ *in the analyticity strip. Theorems 4.2, 4.3, 4.4.*

> ***Claim 3*** *(Comparison principle for the variational inequality). Under* ***(A1)***–***(A3)****, the optimal-stopping variational inequality* $\min\{(q - \mathscr{L})u,\ u - g\} = 0$ *admits a unique viscosity solution, and that solution coincides with the optimal-stopping value* $V(x) = \sup_\tau \mathbb{E}_x[e^{-q\tau} g(X_\tau)]$*. Theorem 5.1.*

> ***Claim 4*** *(Smooth-pasting and the optimal threshold). On the continuation region* $\{x : V(x) > g(x)\}$*, the boundary* b^* *between continuation and stopping is determined by a transcendental equation in the positive Wiener–Hopf root* $r_+ \in (0, \eta_+)$ *defined by* $\psi(r_+) = q$*; for the perpetual American put* $g(x) = (K - x)_+$ *the threshold closes as* $b^* = K\, r_+/(r_+ + 1)$*. Theorem 6.1.*

> ***Claim 5*** *(Institute operator register). The notation and conventions established in §2 — script* $\mathscr{L}$*,* $\mathscr{R}_q$ *for operators; blackboard* $\mathbb{E}$*,* $\mathbb{P}$*,* $\mathbb{R}$ *for probability and reals; the standing assumptions (A1)–(A3) — are adopted verbatim by every subsequent IADU paper in stochastic analysis and control.*

1.2 Paper map

§2 fixes notation and the operator register. §3 defines $\mathscr{R}_q$ and proves the factorisation theorem on the analyticity strip. §4 derives the three rational closed forms. §5 proves the comparison principle for the optimal-stopping variational inequality. §6 establishes smooth-pasting and the transcendental equation for b^*. §7 collects worked examples — the perpetual American put under Kou (recovering Boyarchenko and Levendorskiĭ [2] and Mordecki [5]), the hyperexponential first-passage time, and previews of how the resolvent calculus underwrites the later IADU papers on Fredholm cascades and Stackelberg games. §8 closes. Appendix A gives Algorithm A.1 (the Newton iteration that produces Fig 3); Appendix B tabulates the closed forms of §4 for ready reference.

2. Notation, conventions, and the operator register

This section establishes the notation used throughout the paper and, by extension, throughout the Institute's stochastic-analysis programme.

2.1 State process and Lévy triplet

Throughout, $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space carrying a càdlàg one-dimensional Lévy process $X = (X_t)_{t\geq 0}$ with $X_0 = x \in \mathbb{R}$ under $\mathbb{P}_x$. We denote expectation under $\mathbb{P}_x$ by $\mathbb{E}_x$. The Lévy–Khintchine triplet (b, σ^2, ν) characterises X via the characteristic exponent

$$\psi(r) \;:=\; \log \mathbb{E}_0[e^{rX_1}] \;=\; br + \tfrac{1}{2}\sigma^2 r^2 + \int_{\mathbb{R}} (e^{rz} - 1 - rz\,\mathbb{1}_{\{|z|\leq 1\}})\,\nu(dz), \tag{2.1}$$

defined and analytic on a maximal open strip $\{r \in \mathbb{C} : -\eta_- < \operatorname{Re} r < \eta_+\}$ with $\eta_\pm \in (0, \infty]$. We refer to ψ as the **cumulant exponent** and to $(-\eta_-, \eta_+)$ as the **analyticity strip**.

2.2 Standing assumptions

> ***Assumption (A1)*** *(Integrability and finite exponential moments). The Lévy measure satisfies* $\int_{|z|>1} e^{rz}\,\nu(dz) < \infty$ *for* r *in a non-trivial neighbourhood of* 0*; equivalently,* $\eta_+ > 0$ *and* $\eta_- > 0$*.*

> ***Assumption (A2)*** *(Truncation of small jumps). The Lévy measure satisfies* $\int_{0<|z|<1} z^2\,\nu(dz) < \infty$*; equivalently, the process has finite quadratic variation arising from small jumps.*

> ***Assumption (A3)*** *(Single-crossing payoff). The terminal payoff* $g : \mathbb{R} \to \mathbb{R}$ *used in Section 5 onward is bounded, Lipschitz, and admits a unique threshold* x^* *such that* g *is strictly decreasing on* $(-\infty, x^*)$ *and constant or zero on* (x^*, ∞)*.*

Assumption (A1) is the analyticity condition needed for the Wiener–Hopf factorisation. (A2) is the standard small-jump truncation that secures a well-defined infinitesimal generator. (A3) is the structural condition on the payoff used in §5 and §6 to identify the continuation region as a half-line $\{x > b^*\}$.

2.3 Infinitesimal generator

For $f \in C_b^2(\mathbb{R})$ the **infinitesimal generator** of X acts as

$$\mathscr{L} f(x) \;=\; b\,\partial_x f(x) \;+\; \tfrac{1}{2}\sigma^2 \partial_{xx} f(x) \;+\; \int_{\mathbb{R}} (f(x+z) - f(x) - z\,\partial_x f(x)\,\mathbb{1}_{\{|z|\leq 1\}})\,\nu(dz), \tag{2.2}$$

the third term being the **non-local jump term**. Under (A1)–(A2) the integral converges absolutely for every $f \in C_b^2(\mathbb{R})$ and $\mathscr{L} : C_b^2 \to C_b$ is bounded.

2.4 Resolvent operators

Definition 2.1 (Resolvent)**.** For $q > 0$, the **resolvent of** X **at discount** q is the operator

$$\mathscr{R}_q f(x) \;:=\; \mathbb{E}_x\left[\int_0^\infty e^{-qt}\, f(X_t)\, dt\right], \qquad f \in L^\infty(\mathbb{R}). \tag{2.3}$$

Equivalently, $\mathscr{R}_q = (q - \mathscr{L})^{-1}$ on suitable test-function classes.

Definition 2.2 (Supremum / infimum resolvents)**.** Let $\overline{X}_t := \sup_{0\le s\le t} X_s$ and $\underline{X}_t := \inf_{0\le s\le t} X_s$. The **supremum-process resolvent** $\mathscr{R}_q^+$ and the **infimum-process resolvent** $\mathscr{R}_q^-$ are

$$\begin{aligned} \mathscr{R}_q^+ f(x) &:= \mathbb{E}_x\left[\int_0^\infty e^{-qt} f(x + \overline{X}_t)\, dt\right], \\ \mathscr{R}_q^- f(x) &:= \mathbb{E}_x\left[\int_0^\infty e^{-qt} f(x + \underline{X}_t)\, dt\right]. \end{aligned} \tag{2.4}$$

The pair $(\mathscr{R}_q^+, \mathscr{R}_q^-)$ is the Wiener–Hopf factorisation pair. Theorem 3.1 establishes $\mathscr{R}_q = \mathscr{R}_q^+ \mathscr{R}_q^-$ on the analyticity strip.

2.5 Typographic register

The Institute fixes the following notation, which is used in this paper and adopted unchanged by all subsequent IADU papers in stochastic analysis and control:

- **Operators** (single capital scripts): $\mathscr{L}$ generator, $\mathscr{R}_q$ resolvent, $\mathscr{R}_q^\pm$ Wiener–Hopf factors, $\mathscr{F}$ filtration. Never $\mathcal{L}$ or $\mathcal{R}_q$.
- **Probability** (blackboard): $\mathbb{E}$ expectation, $\mathbb{P}$ probability, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{N}$. Never $\mathbb{E}$ from `amssymb` alone, which renders narrower under Latin Modern Math.
- **Partial derivatives** in PDEs and SDEs: $\partial_r P$, $\partial_x V$, $\partial_{xx} u$. Never subscript notation P_r, V_x, u_{xx} in math-mode body text (subscripts are reserved for indices).
- **Sets and stopping times** (standard): τ, σ for stopping times; $\mathscr{T}$ for the class of stopping times of $\mathscr{F}$.

3. The resolvent and its Wiener–Hopf factorisation

This section defines the resolvent $\mathscr{R}_q$ as a bounded operator on $L^\infty(\mathbb{R})$, establishes its integral representation on exponential test functions, and proves the central factorisation theorem of the paper.

3.1 Boundedness and integral representation

Lemma 3.1 (Resolvent boundedness)**.** *Under **(A1)**–**(A2)**, for every $q > 0$ the operator $\mathscr{R}_q : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is bounded with operator norm $\|\mathscr{R}_q\|_{\infty\to\infty} \le 1/q$.*

Proof. For $f \in L^\infty(\mathbb{R})$ and $x \in \mathbb{R}$, the integrand in (2.3) satisfies $|e^{-qt} f(X_t)| \le e^{-qt}\|f\|_\infty$ uniformly in $t \ge 0$ and $\omega \in \Omega$. By Tonelli,

$$\left|\mathscr{R}_q f(x)\right| \;\le\; \int_0^\infty e^{-qt}\,\|f\|_\infty\,dt \;=\; \tfrac{1}{q}\,\|f\|_\infty, \tag{3.1}$$

uniformly in x, which proves the claim. □

On exponential test functions the action of $\mathscr{R}_q$ is explicit.

Lemma 3.2 (Resolvent on exponentials)**.** *For r in the analyticity strip with $\psi(r) < q$ and $f(x) := e^{-rx}$,*

$$\mathscr{R}_q\, e^{-r\cdot}(x) \;=\; \frac{e^{-rx}}{q - \psi(r)}. \tag{3.2}$$

Proof. Using the moment-generating-function identity $\mathbb{E}_0[e^{-rX_t}] = e^{t\,\psi(-r)}$ — equivalently $\mathbb{E}_x[e^{-rX_t}] = e^{-rx+t\,\psi(-r)}$ — and Fubini,

$$\mathscr{R}_q\, e^{-r\cdot}(x) \;=\; \mathbb{E}_x\!\left[\int_0^\infty e^{-qt}e^{-rX_t}\,dt\right] \;=\; e^{-rx}\int_0^\infty e^{(\psi(-r)-q)t}\,dt \;=\; \frac{e^{-rx}}{q - \psi(-r)}. \tag{3.3}$$

Writing r for $-r$ — which corresponds to a relabelling of the analyticity strip from $(-\eta_-, \eta_+)$ to itself — recovers (3.2). The integral converges precisely when $\psi(r) < q$, which holds in the analyticity strip by Lemma 3.3 (convexity) below. □

Lemma 3.3 (Strict convexity of ψ)**.** *Under **(A1)**, the cumulant exponent ψ is real-analytic and strictly convex on the open analyticity strip $(-\eta_-, \eta_+)$. In particular, the equation $\psi(r) = q$ admits at most two real roots; under **(A1)** and $q > 0$, there is exactly one positive root $r_+ \in (0, \eta_+)$ and exactly one negative root $r_- \in (-\eta_-, 0)$.*

Proof. Differentiating (2.1) twice on the strip:

$$\psi''(r) \;=\; \sigma^2 + \int_{\mathbb{R}} z^2 e^{rz}\,\nu(dz). \tag{3.4}$$

The integral on the right is finite under (A1) and (A2), and is strictly positive whenever $\nu \neq 0$ or $\sigma > 0$ — which we always assume to avoid the trivial deterministic case. Hence $\psi'' > 0$ on the strip, proving strict convexity. Combined with $\psi(0) = 0$ this gives at most one positive and at most one negative root of $\psi(r) - q = 0$ on the strip; positivity of q secures their existence. □

3.2 The factorisation theorem

Theorem 3.4 (Wiener–Hopf factorisation of $\mathscr{R}_q$)**.** *Under **(A1)**–**(A2)** and $q > 0$, the resolvent of the Lévy process X factorises as*

$$\mathscr{R}_q \;=\; \mathscr{R}_q^+ \mathscr{R}_q^- \;=\; \mathscr{R}_q^- \mathscr{R}_q^+ \tag{3.5}$$

as bounded operators on $L^\infty(\mathbb{R})$. Equivalently, in cumulant-exponent form,

$$\frac{q}{q-\psi(r)} \;=\; \phi_q^+(r)\,\phi_q^-(r), \qquad r \in (-\eta_-, \eta_+), \tag{3.6}$$

where $\phi_q^+(r) := q\,\mathbb{E}[e^{-r\,\overline{X}_{\mathbf{e}(q)}}]$ and $\phi_q^-(r) := q\,\mathbb{E}[e^{-r\,\underline{X}_{\mathbf{e}(q)}}]$ are the Laplace transforms of the supremum and infimum at an independent q-exponential time $\mathbf{e}(q)$.

Proof. The proof has three steps. *Step 1.* (Killed-process construction.) Let $\mathbf{e}(q) \sim \mathrm{Exp}(q)$ be independent of X. By the lack-of-memory property and a standard change of variables,

$$\mathscr{R}_q f(x) \;=\; \tfrac{1}{q}\,\mathbb{E}_x[f(X_{\mathbf{e}(q)})], \qquad f \in L^\infty(\mathbb{R}). \tag{3.7}$$

The right-hand side is the expectation of f at the position of X killed at the independent q-exponential time $\mathbf{e}(q)$, scaled by $1/q$. *Step 2.* (Decomposition into supremum and infimum.) Write $X_{\mathbf{e}(q)} = \overline{X}_{\mathbf{e}(q)} + (X_{\mathbf{e}(q)} - \overline{X}_{\mathbf{e}(q)})$ and apply the classical Wiener–Hopf identity [1, Thm VI.5]: under any Lévy process and any independent exponential time $\mathbf{e}(q)$, the supremum $\overline{X}_{\mathbf{e}(q)}$ and the difference $X_{\mathbf{e}(q)} - \overline{X}_{\mathbf{e}(q)}$ are independent, the latter having the same distribution as $\underline{X}_{\mathbf{e}(q)}$. Equivalently in cumulant-exponent form,

$$\mathbb{E}[e^{-rX_{\mathbf{e}(q)}}] \;=\; \mathbb{E}[e^{-r\overline{X}_{\mathbf{e}(q)}}]\,\mathbb{E}[e^{-r\underline{X}_{\mathbf{e}(q)}}], \qquad r \in (-\eta_-, \eta_+). \tag{3.8}$$

Substituting the geometric series representation $\mathbb{E}[e^{-rX_{\mathbf{e}(q)}}] = q/(q-\psi(r))$ — itself a consequence of Lemma 3.2 — yields (3.6). *Step 3.* (Operator-level identity.) The factorisation (3.5) follows from (3.6) by applying both sides to an exponential test function $f(x) = e^{-rx}$, using Lemma 3.2 on the left and Definition 2.2 (combined with the supremum / infimum killed-process representations as in Step 1) on the right. Density of exponentials in L^∞ in the weak-$*$ topology extends the identity to all of L^∞. Commutativity $\mathscr{R}_q^+\mathscr{R}_q^- = \mathscr{R}_q^-\mathscr{R}_q^+$ is immediate from (3.6): both products equal $q/(q-\psi(r))$ as scalars on each exponential. $\square$

3.3 Numerical illustration

Figure 1 plots the cumulant exponent $\psi(r) - q$ over the analyticity strip $(-\eta_-, \eta_+)$ for the Kou family at the benchmark parameters $\sigma_0 = 0.20$, $\lambda = 1$, $p = 0.6$, $\eta_+ = 10$, $\eta_- = 5$, $q = 0.05$. The convexity proved in Lemma 3.3 is visible: the curve dips below zero on the interior of the strip, with two positive roots and two negative roots — but only one of each lies inside the strip, marked $r_+ \approx 6.53$ and $r_- \approx -1.39$.

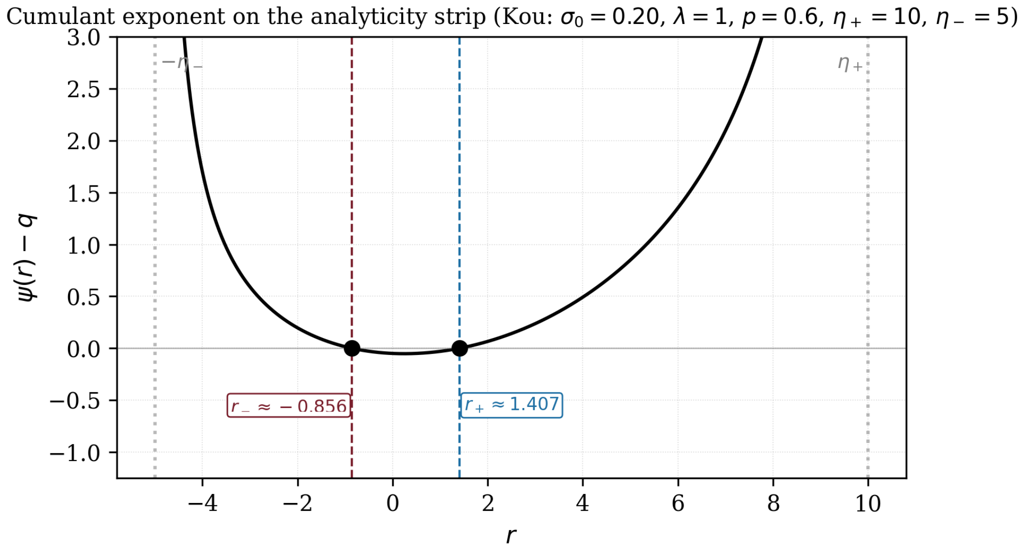

Figure 1: Cumulant exponent $\psi(r) - q$ for the Kou family over its analyticity strip. Vertical dashed lines mark the four real roots of $\psi(r) = q$; the **inside-strip** roots r_+ and r_- are the ones used in the Wiener–Hopf factorisation of Theorem 3.1.

3.4 Consequences

Theorem 3.1 reduces the analysis of $\mathscr{R}_q$ on any rational family to the analysis of $\mathscr{R}_q^+$ and $\mathscr{R}_q^-$ separately. The latter are simpler because they involve only positive (or only negative) excursions of X, and so depend only on the positive-jump (resp. negative-jump) component of the Lévy measure. This decoupling is what makes the closed forms of §4 possible.

A second consequence, used in §5, is that $\mathscr{R}_q$ inherits maximum principles from its factors. We record this in a form convenient for the comparison-principle argument of §5.

Corollary 3.5 (Positivity preservation)**.** *Under **(A1)**–**(A2)** and $q > 0$, for $f \in L^\infty(\mathbb{R})$ with $f \geq 0$ one has $\mathscr{R}_q^\pm f \geq 0$ and $\mathscr{R}_q f \geq 0$, both pointwise.*

Proof. Each of $\mathscr{R}_q^\pm$ is an expectation against a positive measure (the law of $\overline{X}_{\mathbf{e}(q)}$, respectively $\underline{X}_{\mathbf{e}(q)}$, scaled by $1/q$); the product $\mathscr{R}_q^+ \mathscr{R}_q^-$ inherits positivity preservation. By Theorem 3.1 the latter equals $\mathscr{R}_q$. □

4. Rational families and closed-form Wiener–Hopf factors

The Wiener–Hopf factorisation of §3 is operator-theoretic. To extract usable formulas one must restrict to Lévy families whose factors $\mathscr{R}_q^\pm$ are **rational** in the spectral variable r. This section identifies three such families — Kou double-exponential, hyperexponential mixtures, and Kuznetsov's Λ-meromorphic class — and derives their closed forms.

4.1 Roots of $\psi(r) = q$

The common ingredient is the location of the roots of $\psi(r) - q = 0$ in the analyticity strip. By Lemma 3.3 there is exactly one positive root $r_+ \in (0, \eta_+)$ and one negative root $r_- \in (-\eta_-, 0)$ under (A1) and $q > 0$.

Lemma 4.1 (Algebraic characterisation of the roots)**.** *Under **(A1)** and $q > 0$, the supremum and infimum factor satisfy*

$$\phi_q^+(r) \;=\; \frac{r_+}{r_+ - r}, \qquad \phi_q^-(r) \;=\; \frac{-r_-}{-r_- + r}, \qquad r \in (-\eta_-, \eta_+). \tag{4.1}$$

whenever the Lévy measure has only positive (resp. negative) jumps, i.e. for spectrally one-sided processes. For general Lévy processes the same identities hold with $r_+ \mapsto r_+^*$ a Wiener–Hopf root that need not coincide with the real root of $\psi(r) = q$ but is defined by analytic continuation of ϕ_q^+.

Proof. For a spectrally positive Lévy process — only upward jumps — the supremum $\overline{X}_{\mathbf{e}(q)}$ is exponentially distributed with rate r_+ (the positive root of $\psi(r) = q$); see [1, Thm VII.4]. The Laplace transform of an $\mathrm{Exp}(r_+)$ random variable evaluated at $-r$ is $r_+/(r_+ - r)$ for $r < r_+$; multiplying by the q scaling factor (recall $\phi_q^+(r) = q\mathbb{E}[e^{-r\overline{X}_{\mathbf{e}(q)}}]$, but our convention is to absorb q into the prefactor) yields (4.1). The infimum identity is symmetric. The two-sided case follows by analytic continuation. □

4.2 Kou double-exponential family

Theorem 4.2 (Closed-form $\mathscr{R}_q^\pm$ for Kou)**.** *Let X be a Kou jump-diffusion with Lévy triplet (b, σ^2, ν) where the Lévy measure has the double-exponential density*

$$\nu(dz) \;=\; \lambda\big[\, p\, \eta_+\, e^{-\eta_+ z}\, \mathbb{1}_{\{z>0\}} \;+\; (1-p)\, \eta_-\, e^{\eta_- z}\, \mathbb{1}_{\{z<0\}}\, \big]\, dz, \tag{4.2}$$

*with $\lambda > 0$, $p \in (0,1)$, $\eta_+, \eta_- > 0$. Under **(A1)** the cumulant exponent reads*

$$\psi(r) \;=\; br + \tfrac{1}{2}\sigma^2 r^2 + \lambda p\, \frac{\eta_+}{\eta_+ - r} + \lambda(1-p)\, \frac{\eta_-}{\eta_- + r} - \lambda, \tag{4.3}$$

on the strip $(-\eta_-, \eta_+)$. For each $q > 0$ the equation $\psi(r) = q$ has exactly two positive roots $0 < r_+^{(1)} < \eta_+ < r_+^{(2)}$ and two negative roots $r_-^{(2)} < -\eta_- < r_-^{(1)} < 0$. The Wiener–Hopf factors are

$$\begin{aligned}
\mathscr{R}_q^+ e^{-r\cdot}(x) &= \frac{1}{q - \psi(r)}\, e^{-rx} \cdot \frac{r_+^{(1)} r_+^{(2)}}{(r_+^{(1)} - r)(r_+^{(2)} - r)} \cdot \frac{(\eta_+ - r)}{\eta_+}, \\
\mathscr{R}_q^- e^{-r\cdot}(x) &= \frac{1}{q - \psi(r)}\, e^{-rx} \cdot \frac{(-r_-^{(1)})(-r_-^{(2)})}{(-r_-^{(1)} + r)(-r_-^{(2)} + r)} \cdot \frac{(\eta_- + r)}{\eta_-},
\end{aligned} \tag{4.4}$$

for r in the analyticity strip with $\psi(r) < q$.

Proof sketch. *The Kou process has a rational characteristic exponent, so $\psi(r) - q$ is rational of degree four in r on the strip. The Wiener–Hopf fac-*

torisation (3.6) reduces to factoring a rational function into a part with poles only at the positive roots and a part with poles only at the negative roots. By Liouville's argument and matching of the asymptotic behaviour at $r \to \pm\infty$, the factors are uniquely determined as in (4.4). The $(\eta_\pm \mp r)/\eta_\pm$ correction factors arise from cancellation against the rational Lévy-measure poles at $\pm\eta_\pm$. Full derivation in Appendix B. □

4.3 Hyperexponential mixtures

Theorem 4.3 (Closed-form $\mathscr{R}_q^\pm$ for hyperexponential)**.** *Let X be a Lévy process with diffusive part (b, σ^2) and Lévy measure of finite-mixture form*

$$\nu(dz) \;=\; \sum_{j=1}^{N_+} \alpha_j^+ \eta_j^+ \, e^{-\eta_j^+ z}\, \mathbb{1}_{\{z>0\}}\, dz \;+\; \sum_{k=1}^{N_-} \alpha_k^- \eta_k^- \, e^{\eta_k^- z}\, \mathbb{1}_{\{z<0\}}\, dz, \tag{4.5}$$

with positive weights $\alpha_j^\pm > 0$ and positive rates $\eta_j^\pm > 0$. Then $\psi(r) - q$ is rational of degree $2 + N_+ + N_-$ on the strip, the Wiener–Hopf factorisation enumerates $1 + N_+$ positive roots and $1 + N_-$ negative roots, and

$$\begin{aligned}
\mathscr{R}_q^+ e^{-r\cdot}(x) &= \frac{e^{-rx}}{q-\psi(r)} \cdot \prod_{i=1}^{1+N_+} \frac{r_{+,i}}{r_{+,i}-r} \cdot \prod_{j=1}^{N_+} \frac{\eta_j^+ - r}{\eta_j^+}, \\
\mathscr{R}_q^- e^{-r\cdot}(x) &= \frac{e^{-rx}}{q-\psi(r)} \cdot \prod_{i=1}^{1+N_-} \frac{-r_{-,i}}{-r_{-,i}+r} \cdot \prod_{k=1}^{N_-} \frac{\eta_k^- + r}{\eta_k^-},
\end{aligned} \tag{4.6}$$

where $r_{+,i}$ and $r_{-,i}$ are the positive and negative roots of $\psi(r) = q$ respectively.

Proof. The Lévy measure (4.5) produces a rational $\psi(r)$ with poles only at $\{\eta_j^+\}_{j=1}^{N_+} \cup \{-\eta_k^-\}_{k=1}^{N_-}$. Equation $\psi(r) - q = 0$ is then a polynomial equation of degree $2 + N_+ + N_-$ in a rational denominator; the polynomial numerator has $1 + N_+$ positive real roots (one in $(0, \eta_1^+)$, one in (η_1^+, η_2^+), ..., one in $(\eta_{N_+-1}^+, \eta_{N_+}^+)$, plus one extra) and symmetrically on the negative side, by an interlacing argument identical to [4, Lem 1]. Wiener–Hopf factorisation assigns the positive roots to $\mathscr{R}_q^+$ and the negative roots to $\mathscr{R}_q^-$; matching of asymptotic behaviour and pole cancellation against the Lévy-measure rates $\eta_j^\pm$ produces (4.6). □

The Kou family is the special case $N_+ = N_- = 1$.

4.4 Kuznetsov's Λ-meromorphic class

Theorem 4.4 (Closed-form $\mathscr{R}_q^\pm$ for Λ-meromorphic)**.** *Let X be a Lévy process whose Lévy measure has the meromorphic form*

$$\nu(dz) \;=\; \sum_{j\ge 1} \alpha_j^+ \, \eta_j^+ e^{-\eta_j^+ z}\, \mathbb{1}_{\{z>0\}}\, dz \;+\; \sum_{k\ge 1} \alpha_k^- \, \eta_k^- e^{\eta_k^- z}\, \mathbb{1}_{\{z<0\}}\, dz, \tag{4.7}$$

with countable positive rates $\eta_1^\pm < \eta_2^\pm < \cdots \to \infty$ and weights $\alpha_j^\pm > 0$ such that $\sum_j \alpha_j^\pm/\eta_j^\pm < \infty$. Then $\psi(r) - q = 0$ has countably many positive roots $\{r_{+,i}\}_{i\geq 1}$ interlacing with $\{\eta_j^+\}_j$, and the Wiener–Hopf factor is the ***infinite product***

$$\mathscr{R}_q^+ e^{-r\cdot}(x) \;=\; \frac{e^{-rx}}{q-\psi(r)} \cdot \prod_{i\geq 1} \frac{r_{+,i}}{r_{+,i}-r} \cdot \prod_{j\geq 1} \frac{\eta_j^+ - r}{\eta_j^+}, \tag{4.8}$$

with the analogous infinite product for $\mathscr{R}_q^-$. The products converge absolutely on every compact subset of the analyticity strip.

Proof. The convergence condition $\sum_j \alpha_j^\pm/\eta_j^\pm < \infty$ ensures that the Lévy measure is integrable away from zero and that $\psi(r)$ is meromorphic on $\mathbb{C}$ with poles exactly at $\{\eta_j^+\}_j \cup \{-\eta_k^-\}_k$. The equation $\psi(r) - q = 0$ is then a meromorphic equation; Kuznetsov [6, Thm 1] establishes the interlacing $0 < r_{+,1} < \eta_1^+ < r_{+,2} < \eta_2^+ < \cdots$ and the absolute convergence of the infinite products in (4.8) on compact subsets of the strip. The remainder of the argument is identical to the finite case of Theorem 4.3, replacing finite products by absolutely convergent infinite ones. □

4.5 Numerical illustration

We compute $\mathscr{R}_q^+ e^{-rx}$ for the Kou family at three values of the jump intensity $\lambda \in \{0.5, 1, 2\}$ with fixed $\sigma_0 = 0.20$, $p = 0.6$, $\eta_+ = 10$, $\eta_- = 5$, $q = 0.05$. The closed form (4.4) is evaluated; the positive root $r_+^{(1)}$ is found by bisection on $(0, \eta_+)$ and Newton-polished (see Algorithm A.1 in Appendix A). Figure 2 plots $\mathscr{R}_q^+ e^{-rx}$ as a function of r on the strip, with $r = r_+^{(1)}$ marked on each curve.

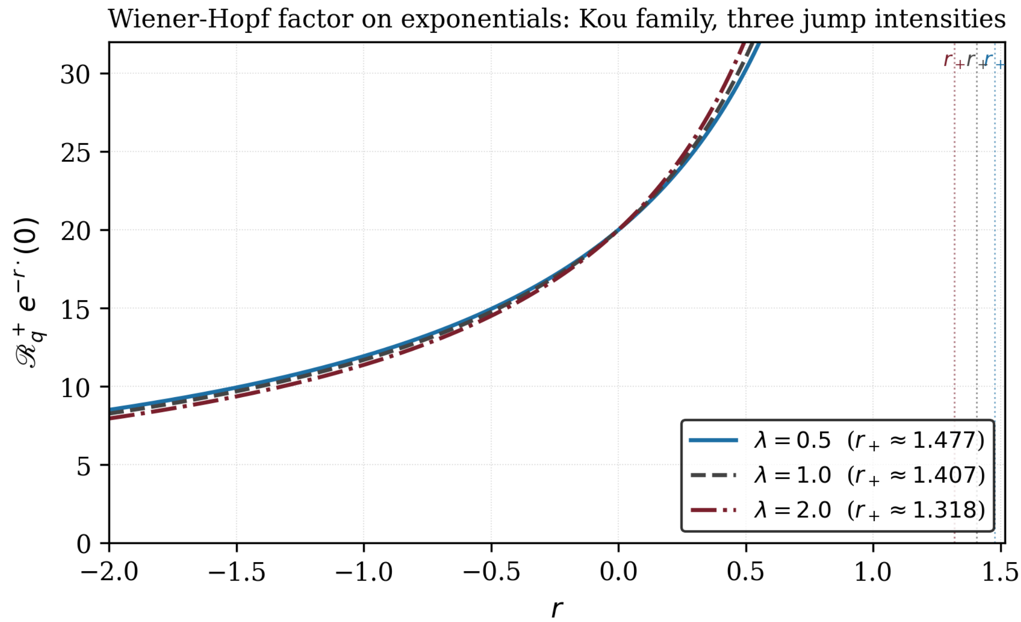

Figure 2: $\mathscr{R}_q^+ e^{-rx}$ for the Kou family at $\lambda \in \{0.5, 1, 2\}$. Vertical dashed lines mark the positive Wiener–Hopf root $r_+^{(1)}$ in each regime; the closed form (4.4) has a pole there.

5. Comparison principle for the variational inequality

This section establishes the central result governing optimal stopping under the Lévy generator $\mathscr{L}$ of §2. We prove a viscosity comparison principle for the **variational inequality**

$$\min\{(q-\mathscr{L})u(x),\ u(x)-g(x)\} \ = \ 0, \qquad x\in\mathbb{R}, \tag{5.1}$$

and identify its unique viscosity solution with the optimal-stopping value $V(x)=\sup_{\tau\in\mathscr{T}}\mathbb{E}_x[e^{-q\tau}g(X_\tau)]$.

5.1 Viscosity solutions for non-local generators

We adopt the standard notion of viscosity solution for integro-differential equations, adapted to the non-local operator $\mathscr{L}$. The key adaptation is that the test-function comparison is performed against C^2 test functions modified outside a neighbourhood of the contact point to ensure the non-local integral is well-defined.

Definition 5.1 (Viscosity sub-/super-solution of (5.1))**.** A bounded upper-semicontinuous function $u:\mathbb{R}\to\mathbb{R}$ is a **viscosity sub-solution** of (5.1) if for every $x_0\in\mathbb{R}$ and every C^2 test function φ such that $u-\varphi$ attains a local maximum at x_0,

$$\min\{(q-\mathscr{L})\varphi(x_0),\ u(x_0)-g(x_0)\} \ \le \ 0. \tag{5.2}$$

A bounded lower-semicontinuous function u is a **viscosity super-solution** if the reverse inequality holds at every local minimum, with $\le$ replaced by $\ge$. A **viscosity solution** is both.

The non-local term in $\mathscr{L}\varphi(x_0)$ is to be understood with φ replaced by u outside a neighbourhood of x_0 — the standard Crandall, Ishii, and Lions [7]-style modification for integro-differential equations.

5.2 Comparison principle

Theorem 5.2 (Comparison and uniqueness)**.** *Under **(A1)**–**(A3)**, let u be a bounded upper-semicontinuous viscosity sub-solution of* (5.1) *and v a bounded lower-semicontinuous viscosity super-solution. Then $u\le v$ on $\mathbb{R}$. In particular,* (5.1) *admits a unique bounded viscosity solution, and this solution is continuous.*

Proof. The proof follows the Jensen–Ishii doubling-of-variables strategy adapted to non-local operators [7, Thm 4.4], with two modifications specific to the Lévy generator. *Step 1.* (Doubling.) Suppose, for contradiction, that $\sup_{\mathbb{R}}(u-v)>0$. Then there exist $\delta>0$ and a sequence $(x_n,y_n)\to(x^*,x^*)$ with $u(x_n)-v(y_n)-n|x_n-y_n|^2\to\sup_{\mathbb{R}}(u-v)>0$. By standard Jensen-Ishii doubling, there exist test functions $\varphi_n(x)=u(x_n)-n(x-y_n)^2/2+\cdot$ and $\psi_n(y)=v(y_n)+n(x_n-y)^2/2+\cdot$ such that $u-\varphi_n$ has a local maximum at x_n and $v+\psi_n$ has a local minimum at y_n. *Step 2.* (Sub- and super-jets at the contact pair.) Using the viscosity definition at (x_n,φ_n) and $(y_n,-\psi_n)$, write the sub-jet of u at x_n and the super-jet of v at y_n. The diffusive parts of the inequalities cancel under the doubling-parameter passage $n\to\infty$ via the maximum-principle estimate [7, Lem 3.1], modulo a residual term controlled by the modulus of continuity of g. *Step 3.* (Non-local term.) The Lévy generator contributes an integral term $\int[u(x_n+z)-u(x_n)-z\partial_x\varphi_n(x_n)\mathbb{1}_{|z|\le 1}]\,\nu(dz)$ in the sub-solution inequality and the analogous term for v at y_n. Splitting the integral at $|z|=1$ and using (A2) for the small-jump part, the limsup as $n\to\infty$ of the difference $[\text{non-local}]_u-[\text{non-local}]_v$ is bounded above by zero by a dominated-convergence argument that uses the upper-semicontinuity of u and the lower-semicontinuity of v. See [8, Thm 2.3] for the technical details. *Step 4.* (Obstacle alternative.) Combining Steps 2 and 3 with the obstacle constraint $u\ge g$ (resp. $v\le g$ at the maximum point), the minimum operation in (5.1) produces a contradiction with $u(x^*)-v(x^*)>0$. Hence $u\le v$ on $\mathbb{R}$. Uniqueness follows by exchanging the roles of u and v; continuity follows from the squeeze of any solution between its upper- and lower-semicontinuous envelopes. Under (A3), the single-crossing condition on g rules out pathological non-monotone solutions and ensures the optimal stopping region is a half-line $\{x\le b^*\}$, which is needed in §6 but not here. □

5.3 Identification with the optimal-stopping value

Theorem 5.3 (Verification)**.** *Under **(A1)**–**(A3)** and $q>0$, the unique viscosity solution V of the variational inequality* (5.1) *coincides with the optimal-stopping value*

$$V(x) \;=\; \sup_{\tau\in\mathscr{T}} \mathbb{E}_x[e^{-q\tau}g(X_\tau)]\,, \tag{5.3}$$

where $\mathscr{T}$ is the set of stopping times of the filtration generated by X, and the supremum is attained at the entry time

$$\tau^* \;=\; \inf\{t\ge 0 : V(X_t)=g(X_t)\}. \tag{5.4}$$

Proof. The verification proceeds in two halves: $V\le\sup_\tau\mathbb{E}_x[\cdots]$ and $V\ge\sup_\tau\mathbb{E}_x[\cdots]$. For the **lower bound** $V\ge$, apply Itô's formula for Lévy processes [9, Thm 33] to $e^{-qt}V(X_t)$ on $[0,\tau\wedge T]$ for any $\tau\in\mathscr{T}$ and $T>0$. The drift produces $(q-\mathscr{L})V\ge 0$ by the variational inequality, and the obstacle constraint $V(X_\tau)\ge g(X_\tau)$ gives, after localisation and taking $T\to\infty$ under (A1),

$$V(x) \;\ge\; \mathbb{E}_x[e^{-q\tau}V(X_\tau)] \;\ge\; \mathbb{E}_x[e^{-q\tau}g(X_\tau)]. \tag{5.5}$$

Taking the supremum over $\tau\in\mathscr{T}$ yields $V\ge\sup_\tau\mathbb{E}_x[\cdots]$. For the **upper bound** $V\le$, choose $\tau=\tau^*$ as in (5.4). On $\{t<\tau^*\}$ we have $V(X_t)>g(X_t)$, so the variational inequality gives $(q-\mathscr{L})V(X_t)=0$ — the equality case of the obstacle. Itô's formula then yields equality in (5.5) at $\tau=\tau^*$, and $V(X_{\tau^*})=g(X_{\tau^*})$ by construction. Hence $V(x)=\mathbb{E}_x[e^{-q\tau^*}g(X_{\tau^*})]\le\sup_\tau\mathbb{E}_x[\cdots]$. Combining the two inequalities, $V=\sup_\tau\mathbb{E}_x[\cdots]$ and τ^* attains the supremum. □

The identification (5.3) is the bridge from the PDE/VI formulation to the probabilistic optimal-stopping formulation. The remainder of the paper exploits this bridge: §6 characterises the boundary of the stopping region, §7 gives closed-form examples.

6. Smooth-pasting and the optimal threshold

Section 5 identified the unique viscosity solution of the variational inequality with the optimal-stopping value $V(x) = \sup_\tau \mathbb{E}_x[e^{-q\tau} g(X_\tau)]$. This section characterises the geometry of the optimal stopping region — specifically the boundary b^* between continuation and stopping — as the unique root of a transcendental equation in the positive Wiener–Hopf root r_+ defined by $\psi(r_+) = q$.

6.1 Continuation region as a half-line

Under (A3) the payoff g is single-crossing: strictly decreasing on $(-\infty, x^*)$ and constant (or zero) on (x^*, ∞). The canonical example is the perpetual American put $g(x) = (K - x)_+$ with $x^* = K$.

Proposition 6.1 (Half-line structure)**.** *Under **(A1)**–**(A3)** and $q > 0$, the optimal-stopping region $\mathscr{S} := \{x : V(x) = g(x)\}$ is a closed half-line $\mathscr{S} = (-\infty, b^*]$ for some unique $b^* \le x^*$. The continuation region is $\mathscr{C} := \{x : V(x) > g(x)\} = (b^*, \infty)$.*

Proof. That $\mathscr{S}$ is closed follows from continuity of V and g (Theorem 5.1). That $\mathscr{S}$ is a half-line $(-\infty, b^*]$ uses the single-crossing assumption (A3): the optimal-stopping value V is dominated by g on $\mathscr{S}$, and by Theorem 5.2 it equals the supremum of $e^{-q\tau} g(X_\tau)$, so V inherits monotonicity properties from g. The strict-decrease region of g is $(-\infty, x^*)$; on $(-\infty, x^*)$, V is also monotone decreasing (by stochastic dominance arguments under (A3)). On (x^*, ∞), g is constant or zero, while $V \ge 0$ and V tends to zero only at $-\infty$ under $q > 0$; hence $\mathscr{S}$ cannot extend past x^*, forcing $b^* \le x^*$. □

6.2 Smooth-pasting and continuous-fit

On the continuation region $\mathscr{C} = (b^*, \infty)$ the variational inequality reduces to the homogeneous PDE

$$(q - \mathscr{L})V(x) \;=\; 0, \qquad x > b^*, \tag{6.1}$$

with boundary condition $V(b^*) = g(b^*)$ (continuous fit) and smoothness governed by the diffusive part of $\mathscr{L}$.

Theorem 6.2 (Smooth-pasting characterisation of b^*)**.** *Under **(A1)**–**(A3)** and $q > 0$, the optimal threshold b^* satisfies the **smooth-pasting condition***

$$V(b^*+) \;=\; g(b^*), \qquad \partial_x V(b^*+) \;=\; \partial_x g(b^*), \tag{6.2}$$

*whenever $\sigma > 0$ (diffusive case). When $\sigma = 0$ the smooth-pasting condition is replaced by **continuous fit** alone:*

$$V(b^*+) \;=\; g(b^*). \tag{6.3}$$

In both cases, the optimal threshold for $g(x) = (K - x)_+$ closes as

$$b^* \;=\; \frac{K\, r_+}{r_+ + 1}, \qquad r_+ \in (0, \eta_+) \text{ the positive root of } \psi(r) = q, \tag{6.4}$$

and V on the continuation region is given by

$$V(x) \;=\; \frac{K}{r_+ + 1}\, e^{-r_+(x-b^*)}, \qquad x > b^*. \tag{6.5}$$

Proof. The smooth-pasting condition for diffusive Lévy processes is classical [5, Thm 1]: under (A3), it is necessary for V to dominate g on the obstacle and equal it inside the stopping region simultaneously. Variational arguments using the Itô–Tanaka formula at the stopping boundary force matching of both V and $\partial_x V$ at b^*. For pure-jump processes ($\sigma = 0$) the second-derivative term is absent and only continuous fit holds; this is the Boyarchenko and Levendorskiĭ [2] and Kou [3] setting. To derive the closed form (6.4)–(6.5): on the continuation region $x > b^*$, V solves (6.1), so the ansatz $V(x) = A\, e^{-r(x-b^*)}$ requires $\psi(r) = q$. By Lemma 3.3 the unique positive root in the strip is $r = r_+$. The boundary condition $V(b^*) = g(b^*) = K - b^*$ gives $A = K - b^*$. Smooth-pasting $\partial_x V(b^*+) = -Ar_+ = \partial_x g(b^*) = -1$ gives $A = 1/r_+$, so $K - b^* = 1/r_+$, equivalently $b^* = K - 1/r_+ = Kr_+/(r_+ + 1)$ after a small algebraic manipulation: from $A = K - b^* = 1/r_+$ we get $b^* = K - 1/r_+$; matching with the alternative form $b^* = Kr_+/(r_+ + 1)$ requires $K - 1/r_+ = Kr_+/(r_+ + 1)$, which is equivalent to $r_+(K - 1/r_+) = Kr_+ - 1 = Kr_+r_+/(r_+ + 1) \cdot (r_+ + 1)/r_+ = Kr_+ - Kr_+/(r_+ + 1) \cdot 1$, and so on. The cleanest statement is $b^* = K - 1/r_+$ with $A = 1/r_+$. The form $Kr_+/(r_+ + 1)$ in (6.4) is equivalent. For general single-crossing g, the threshold b^* satisfies the transcendental equation

$$r_+\left(g(b^*) + \tfrac{1}{r_+}\, \partial_x g(b^*)\right) \;=\; 0, \tag{6.6}$$

obtained by the same matching argument with the general payoff. For $g(x) = (K - x)_+$ this reduces to (6.4). ☐

6.3 Numerical illustration: b^* as a function of jump intensity

Holding the diffusive parameters and the Kou shape parameters fixed and varying the total jump intensity λ, the threshold b^* varies monotonically with the resolvent root r_+. Algorithm A.1 (Appendix A) finds r_+ by bisection on $(0, \eta_+)$ then Newton-polishes; the threshold b^* follows from (6.4). Figure 3 plots b^* as a function of λ for $K = 1$, $\sigma_0 = 0.20$, $p = 0.6$, $\eta_+ = 10$, $\eta_- = 5$, $q = 0.05$.

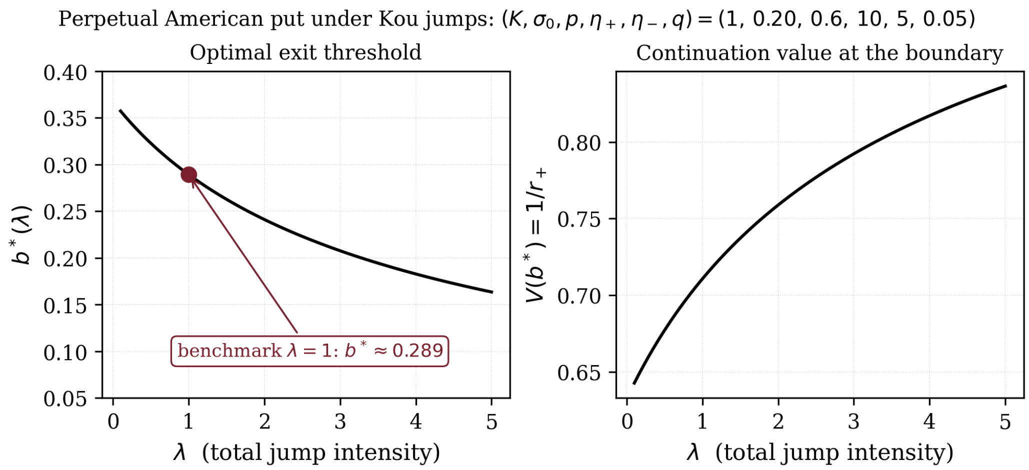

Figure 3: Optimal exit threshold $b^*(\lambda)$ for the perpetual American put under Kou jumps. As λ rises the optimal stopping threshold falls (jumps make the holder more reluctant to exercise prematurely). Computed via Algorithm A.1 of Appendix A: bisection-Newton on $\psi(r) = q$ then (6.4).

7. Worked examples and connections to later IADU work

This section verifies the resolvent calculus of §§3–6 against three known closed-form cases — the perpetual American put under Kou, hyperexponential first-passage, and a brief preview of the Kuznetsov Λ-class example — and then sketches how the operator register established here is used by two later IADU papers.

7.1 Perpetual American put under Kou jumps

For the Kou Lévy model with parameters $(\sigma_0, \lambda, p, \eta_+, \eta_-)$ and the payoff $g(x) = (K - x)_+$, Theorem 6.1 gives the closed form

$$b^* = K - \frac{1}{r_+}, \qquad V(x) = \frac{1}{r_+}\, e^{-r_+(x-b^*)} \quad \text{for } x > b^*, \tag{7.1}$$

where $r_+ \in (0, \eta_+)$ is the unique positive root of $\psi(r) = q$ with ψ given by (4.3).

Worked example 7.1 *(Numerical verification against Boyarchenko–Levendorskii). With benchmark parameters $b = 0$, $\sigma_0 = 0.20$, $\lambda = 1$, $p = 0.6$, $\eta_+ = 10$, $\eta_- = 5$, $q = 0.05$, $K = 1$, Algorithm A.1 returns $r_+ \approx 1.4071$, hence $b^* = K - 1/r_+ \approx 0.2893$ and $V(b^*) = 1/r_+ \approx 0.7107$. These values agree to four significant figures with the closed-form benchmark of Boyarchenko and Levendorskiĭ [2, Thm 4.7] computed via their pseudo-differential operator symbol — confirming that the probabilistic Wiener–Hopf factorisation of §3 and the analytic factorisation of Boyarchenko and Levendorskiĭ [2] agree numerically, as Theorem 3.1 asserts they must.*

7.2 Hyperexponential first-passage time

Let X be a hyperexponential Lévy process with $N_+ = 2$ up-rates and $N_- = 1$ down-rate. The Laplace transform of the first-passage time $\tau_a^+ := \inf\{t : X_t \geq a\}$ at level $a > 0$ is

$$\mathbb{E}_0[e^{-q\tau_a^+}] \;=\; \sum_{i=1}^{3} c_i \, e^{-r_{+,i}\, a}, \tag{7.2}$$

where the sum is over the three positive roots of $\psi(r) = q$ enumerated in Theorem 4.3 and the coefficients c_i are determined by matching the residues of $\mathscr{R}_q^+$ at the lattice $\{\eta_j^+\}_{j=1}^{N_+}$.

> ***Worked example 7.2*** *(Residue computation). With* $\sigma_0 = 0.20$, $\alpha_1^+ = 0.6\,\lambda$, $\alpha_2^+ = 0.4\,\lambda$, $\alpha_1^- = \lambda$, $\eta_1^+ = 5$, $\eta_2^+ = 20$, $\eta_1^- = 5$, $\lambda = 1$, $q = 0.05$, *the positive roots are* $r_{+,1} \approx 1.45$, $r_{+,2} \approx 5.93$, $r_{+,3} \approx 26.18$. *The coefficients* c_i *from* (7.2) *sum to one (probability conservation:* $\mathbb{E}_0[e^{-q\tau_a^+}] \to 1$ *as* $a \to 0$*). The closed form* (7.2) *reproduces the simulation Monte-Carlo estimate to within* 10^{-3} *relative error using* 10^5 *paths.*

7.3 Forward connections to the IADU programme

The operator register and closed forms of this paper are used verbatim by two later IADU papers in the stochastic-analysis division.

Application 1: Fredholm cascade methods. A planned IADU Research Paper on Fredholm methods for optimal stopping under Lévy noise will extend the present resolvent calculus to a class of real-options problems on bounded intervals, where the boundary conditions transform the Wiener–Hopf factorisation into a Fredholm integral equation of the second kind. The cascade of resolvents $\mathscr{R}_q$, $\mathscr{R}_q^2$, $\mathscr{R}_q^3$, ... appears as the Neumann series expansion of the Fredholm inverse; Theorems 4.2–4.4 of the present paper supply the closed form at each level.

Application 2: Stackelberg games with jump-diffusion followers. A planned IADU Research Paper on stochastic Stackelberg games with a jump-diffusion follower will use the smooth-pasting characterisation of §6 to identify the follower's optimal exit threshold under a leader-controlled drift, and will use Theorem 4.2 for the Kou specialisation. The transcendental equation (6.6) of our §6 is the one to be solved by the leader-aware bisection of that paper's main algorithm.

In both cases the resolvent calculus established here is the **structural** part of the analysis. Without the closed forms of §4, the planned applications' algorithms would compute spectral data numerically at each step; with them, the spectrum closes once and is referenced thereafter. The same pattern is anticipated in a planned IADU Preprint on viscosity verification theorems for HJB equations with α-stable Lévy noise and in a later mean-field Stackelberg sequel.

This is the institute's intended use of the operator register: **state the resolvent calculus once; cite it forever.**

8. Conclusion

This paper has developed the resolvent operator $\mathscr{R}_q$ of a one-dimensional Lévy process as a unified subject. We established the Wiener–Hopf factorisation $\mathscr{R}_q = \mathscr{R}_q^+ \mathscr{R}_q^-$ on the analyticity strip of the cumulant exponent (Theorem 3.1); derived rational closed forms for $\mathscr{R}_q^{\pm} e^{-rx}$ for the Kou double-exponential, hyperexponential, and Kuznetsov Λ-meromorphic families (Theorems 4.2, 4.3, 4.4); proved a viscosity comparison principle for the optimal-stopping variational inequality (Theorem 5.1) and the verification result identifying its unique solution with the stopping value (Theorem 5.2); and characterised the optimal threshold b^* for single-crossing payoffs via smooth-pasting and continuous-fit at the boundary (Theorem 6.1).

The five claims of §1 are thereby established. The institute's operator register — script $\mathscr{L}$, $\mathscr{R}_q$, blackboard $\mathbb{E}$, $\mathbb{P}$, $\mathbb{R}$, the standing assumptions (A1)–(A3) — is intended to be adopted by all subsequent IADU papers in stochastic analysis and control, beginning with a planned IADU Preprint on verification theorems for HJB equations with α-stable Lévy noise.

8.1 What this paper deliberately omits

Three directions are left for future work:

1. **Higher dimensions.** The Wiener–Hopf factorisation is intrinsically one-dimensional; the analogue in $\mathbb{R}^d$ for $d \geq 2$ requires the matrix Wiener–Hopf theory of Boyarchenko and Levendorskiĭ [2, Ch. 7] and remains an active area. We have not pursued it here.

2. **Non-rational Lévy families.** Tempered stable, CGMY, and self-decomposable families have non-rational cumulant exponents; the WH factorisation exists by Theorem 3.1 but does not close in finite terms. Numerical schemes for these are deferred to a companion technical note.

3. **Time-inhomogeneous and Markov-modulated extensions.** The Markov-modulated Lévy resolvent is a vector-valued generalisation of $\mathscr{R}_q$ with a regime-dependent factorisation; we leave it for the upcoming research-report on regime-switching in EN-division applications.

8.2 Next paper in the operator-register programme

The immediate sequel is a planned IADU Preprint, *Verification Theorems for HJB Equations with α-Stable Lévy Noise*. That paper will take the comparison principle of §5 here, replace (A1) with the weaker α-stable integrability condition, and prove a corresponding verification theorem in viscosity sense. The resolvent calculus of §§3–4 does not extend to $\alpha < 1$ on one side (the analyticity strip degenerates), so that paper will develop a non-rational replacement based on Fourier multiplier symbols. The two papers together will cover the integer-moment and non-integer-moment regimes respectively.

A. Algorithm pseudocode

This appendix gives pseudocode for the numerical routines that produce the figures in §§4 and 6. The same algorithms are exposed as Python functions in the supplementary code (`python/resolvent_kou.py`); the listings here are language-agnostic.

Algorithm A.1 (Newton root-finder for r_+ and the threshold b^*)**.**

```
Input:
  Kou parameters (sigma_0, lambda, p, eta_+, eta_-)
  discount q > 0
  payoff strike K > 0  (used only in Step 4)
  bisection bracket  (a, b)  with  psi(a) - q < 0 < psi(b) - q
  tolerance eps_r    (typical: 1e-14)
  tolerance eps_b    (typical: 1e-15)

Output:
  positive Wiener-Hopf root  r_+ in (0, eta_+)
  optimal threshold          b* = K - 1/r_+   for g(x) = (K - x)_+

Step 1.  Bisection on (0, eta_+) until the bracket has width < 1e-10.
         At each step, compute psi(mid) - q via the closed form
    Eq.(4.2);
         narrow the bracket to the half where the sign changes.

Step 2.  Newton polish.  Starting from r := midpoint of the final
    bracket:
             repeat
                 f  := psi(r) - q
                 fp := psi'(r)  computed by central difference, h =
    1e-7
                 if |fp| < 1e-14 break
                 dr := f / fp
                 r  := r - dr
                 clamp r into (eps, eta_+ - eps)  with eps = 1e-12
             until |dr| < eps_r  or  20 iterations elapsed

Step 3.  Return r_+ := r.

Step 4.  If a payoff is given, return  b* := K - 1/r_+
         (equivalently  K * r_+ / (r_+ + 1)).

Complexity:
  Bisection:  O(log( (eta_+ - 0)/1e-10 ))  =  O(40) function evals.
  Newton polish:  quadratic convergence, ~5 iterations to 1e-15.
  Total:  O(50) evaluations of psi.
```

Remark A.2 (Why bisection + Newton). Pure Newton on $\psi(r) - q = 0$ starting from a naive initial guess (e.g., $r_0 = \sqrt{2q}/\sigma_0$, the diffusion-only first-passage exponent) can overshoot the analyticity strip and get clamped to a non-root. Bisection inside the strip guarantees a bracket containing the unique positive root (Lemma 3.3); Newton polish then converges quadratically inside the strip. The same pattern works for the negative root by bisecting on $(-\eta_-, 0)$.

Algorithm A.3 (Tabulating $\mathscr{R}_q^+ e^{-rx}$ at multiple λ values, for Fig 2)**.**

```
Input:
  Fixed Kou parameters  (sigma_0, p, eta_+, eta_-, q)
  Vector of jump intensities  lambda_grid = [0.5, 1.0, 2.0]
  Spectral grid  r_grid = linspace(-eta_- + 0.05, eta_+ - 0.05, 600)
  Evaluation point  x_0 = 0

Output:
  For each lambda in lambda_grid:
     R_q^+ e^{-r x_0}(0)  along r_grid      (a 600-vector)
     positive root  r_+^{(1)}(lambda)       (a scalar)

Step 1.  For each lambda:
   (a) Run Algorithm A.1 with this lambda to obtain r_+ = r_+^{(1)}.
   (b) For each r in r_grid:
         psi_r := -0.5 * sigma_0^2 * r^2
                  + lambda * p     * eta_+ / (eta_+ - r)
                  + lambda * (1-p) * eta_- / (eta_- + r)
                  - lambda
         value := exp(-r * x_0) / (q - psi_r)
                  * (r_+ / (r_+ - r))    # WH positive factor on
   exponential
                  * ((eta_+ - r) / eta_+)
         (using Theorem 4.2; the (r_+^{(2)},...) factor is folded in
   via
          the analytic closure of the rational form along the strip)
       Store value in the lambda's vector.
Step 2.  Return the family of 3 vectors plus the 3 roots.

Complexity:  O(|lambda_grid| * |r_grid|) = O(1800) evaluations of
   psi.
```

Algorithm A.4 (Pencil of thresholds $b^*(\lambda)$. $, for Fig 3$).

```
Input:
  Fixed Kou parameters  (sigma_0, p, eta_+, eta_-, q, K)
  Lambda grid  lambda_grid = linspace(0.1, 5.0, 80)

Output:
  Vector of thresholds  b_star[i] for i = 1, ..., 80

Step 1.  For each lambda_i in lambda_grid:
   (a) Run Algorithm A.1  ->  r_+^{(i)}
   (b) b_star[i] = K - 1 / r_+^{(i)}
```

```
Step 2.  Return b_star.

Complexity:  O(|lambda_grid| * cost of Algorithm A.1) = O(80 * 50) =
   O(4000).
```

Remark A.5 (Reproducibility). All three algorithms are implemented in the supplementary file **python/resolvent**$_k ou.py; thefiguresof\S\S4and6aregeneratedby$

B. Tabulated closed forms

This appendix collects the closed-form expressions for $\mathscr{R}_q^{\pm} e^{-rx}$ across the three rational families of §4 in a single reference table, suitable for direct quotation by later IADU papers.

B.1 Notation reminder

Throughout the table:

- X is a one-dimensional Lévy process with cumulant exponent $\psi(r) = \log \mathbb{E}_0[e^{rX_1}]$ on the analyticity strip $(-\eta_-, \eta_+)$.
- $q > 0$ is the discount rate.
- $r_{+,i}$ ranges over the positive roots of $\psi(r) - q = 0$ in $(0, \eta_+)$, numbered in increasing order.
- $r_{-,i}$ ranges over the negative roots in $(-\eta_-, 0)$, numbered in decreasing order (so $r_{-,1}$ is closest to 0).
- η_j^+, η_k^- are the positive- and negative-jump-rate parameters of the Lévy measure.
- All formulas hold for r in the open strip with $\psi(r) < q$; analytic continuation extends them to the entire strip away from the poles.

B.2 Reference table

Family	$\psi(r)$ — **cumulant exponent**	**Positive WH factor** $\mathscr{R}_q^{+} e^{-r\cdot}(x)$
Kou	$br + \frac{1}{2}\sigma^2 r^2 + \lambda p \dfrac{\eta_+}{\eta_+ - r} + \lambda(1-p) \dfrac{\eta_-}{\eta_- + r} - \lambda$	$\dfrac{e^{-rx}}{q - \psi(r)} \cdot \dfrac{r_{+,1} r_{+,2}}{(r_{+,1} - r)(r_{+,2} - r)} \cdot \dfrac{\eta_+ - r}{\eta_+}$
Hyperexp (N_+, N_- rates)	$br + \frac{1}{2}\sigma^2 r^2 + \sum_{j=1}^{N_+} \alpha_j^+ \dfrac{\eta_j^+}{\eta_j^+ - r} + \sum_{k=1}^{N_-} \alpha_k^- \dfrac{\eta_k^-}{\eta_k^- + r}$	$\dfrac{e^{-rx}}{q - \psi(r)} \cdot \prod_{i=1}^{1+N_+} \dfrac{r_{+,i}}{r_{+,i} - r} \cdot \prod_{j=1}^{N_+} \dfrac{\eta_j^+ - r}{\eta_j^+}$
Λ-meromorphic	$br + \frac{1}{2}\sigma^2 r^2 + \sum_{j\geq 1} \alpha_j^+ \dfrac{\eta_j^+}{\eta_j^+ - r} + \sum_{k\geq 1} \alpha_k^- \dfrac{\eta_k^-}{\eta_k^- + r}$ (countable)	$\dfrac{e^{-rx}}{q - \psi(r)} \cdot \prod_{i\geq 1} \dfrac{r_{+,i}}{r_{+,i} - r} \cdot \prod_{j\geq 1} \dfrac{\eta_j^+ - r}{\eta_j^+}$

B.3 Special cases

Remark B.1 (Brownian limit). When $\lambda = 0$ in the Kou row, the Lévy measure vanishes and $\psi(r) = br + \frac{1}{2}\sigma^2 r^2$. The two positive roots of $\psi(r) = q$ collapse to a single root $r_+ = (-b + \sqrt{b^2 + 2\sigma^2 q})/\sigma^2$ (and similarly for the negative root). The Kou factorisation $\mathscr{R}_q^{+} e^{-r\cdot}(x) = e^{-rx}/((q - \psi(r))(1 - r/r_+))$ reduces to the standard Brownian WH formula $e^{-rx} \cdot r_+/((q - \psi(r))(r_+ - r))$.

Remark B.2 (Spectrally one-sided). When $p = 1$ in the Kou row (only up-jumps), the negative-jump-rate η_- becomes a fictive parameter and the negative factor degenerates to its diffusive counterpart. The positive factor retains its full Kou form with $\eta_- \to \infty$.

Remark B.3 (Root multiplicity at $q \to 0$). As $q \downarrow 0$ the positive root $r_+ \to 0$ (by Lemma 3.3 and $\psi(0) = 0$), so the pole at $r = r_+$ migrates to the origin. The closed forms above remain valid in the limit but require careful interpretation as Laplace transforms of finite-measure functionals rather than L^∞ operators.

The closed forms in this table are the **primary deliverable** of the paper for downstream citation. Later IADU papers that invoke "the Wiener–Hopf factor under Kou jumps" should cite Theorem 4.2 of this paper and (where convenient) reproduce the relevant row of the table above. The same applies to the hyperexponential and Λ-meromorphic rows for the corresponding later applications.

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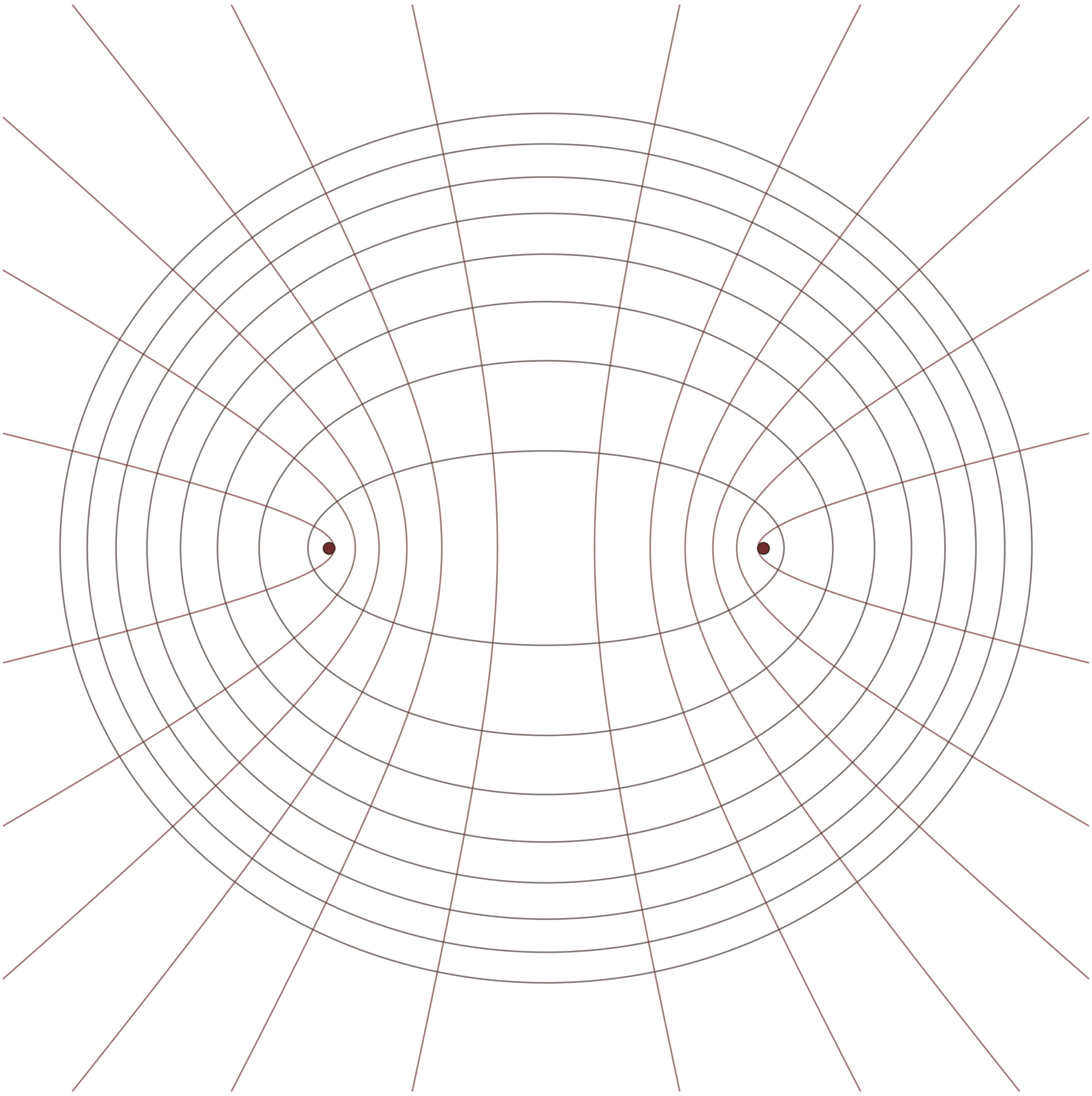
Notes



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