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# Saddle-Point Structures for Two-Player Zero-Sum SDEs with Lévy Noise

## Existence, coincidence of values, and a non-local first-order condition

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# The problem (1/2)

- Two-player zero-sum stochastic differential games model adversarial stochastic environments — central banks against speculators, risk managers against adversarial markets — and the classical Brownian theory is well-developed.
- The generator $\mathscr{L}$ is purely diffusive; value functions satisfy a local Hamilton–Jacobi–Isaacs equation, tractable via viscosity-solution comparison principles.

# The problem (2/2)

- Observe that no existing zero-sum SDG reference treats Lévy jumps as a first-class object — the diffusive $\mathscr{L}$ must be extended, and the value equation becomes integro-differential.
- We assemble the operator and comparison-principle infrastructure of the institute's resolvent paper into a self-contained zero-sum framework for Lévy-driven state processes.

# Setup

Throughout this work, the state $X = (X_t)_{t\geq 0}$ is a càdlàg Lévy process on $(\Omega, \mathscr{F}, \mathbb{P})$, characterised by triplet $(b, \sigma^2, \nu)$ and cumulant exponent $\psi(r) = \log \mathbb{E}_0[e^{rX_1}]$, analytic on $(-\eta_-, \eta_+)$ — the generator $\mathscr{L}$ takes the Lévy–Khintchine form

$$\mathscr{L} f(x) \; = \; b\,\partial_x f(x) + \tfrac{1}{2}\sigma^2\,\partial_{xx} f(x) + \int_{\mathbb{R}} \big(f(x+z) - f(x) - z\,\partial_x f(x)\,\mathbb{1}_{\{|z|\leq 1\}}\big)\,\nu(dz),$$

and standing assumptions **(A1)–(A3)** are those of (Mészáros et al., 2023).

# The central equation

- We work throughout with a càdlàg Lévy process $X = (X_t)_{t\geq 0}$ on $(\Omega, \mathscr{F}, \mathbb{P})$, specified by its Lévy triplet $(b, \sigma^2, \nu)$ and cumulant exponent $\psi(r) = \log \mathbb{E}_0[e^{rX_1}]$ — the latter analytic on the maximal strip $(-\eta_-, \eta_+)$.
- The infinitesimal generator — the central object of the analysis — is

$$\mathscr{L} f(x) \;=\; b\,\partial_x f(x) + \tfrac{1}{2}\sigma^2 \partial_{xx} f(x) + \int_{\mathbb{R}} \big(f(x+z) - f(x) - z\,\partial_x f(x)\,\mathbb{1}_{\{|z|\leq 1\}}\big)\,\nu(dz),$$

with standing assumptions **(A1)–(A3)** as in the institute's resolvent paper (Mészáros et al., 2023).

# Main results (1/3)

- **Value-function regularity.** The upper value $V^+$ and lower value $V^-$ of the zero-sum Lévy SDG are bounded and 1/2-Hölder continuous in $x$ under standing assumptions **(A1)–(A3)** of (Mészáros et al. 2023) and Lipschitz running cost.
- **Coincidence of values.** Under the Isaacs minimax condition

$$\sup_{u \in U} \inf_{v \in V} H(x, p, M, u, v) = \inf_{v \in V} \sup_{u \in U} H(x, p, M, u, v), \tag{1}$$

the values coincide and define the unique viscosity solution of the HJI integro-differential equation $\partial_t V + H(x, \partial_x V, \partial_{xx} V) + \mathscr{I}[V] = 0$.

## Main results (2/3)

- **Saddle-point first-order condition.** At every state in the interior of the continuation region, the saddle pair $(u^*, v^*) \in U \times V$ satisfies a **non-local** first-order condition

$$\partial_u H(x, \nabla V, u^*, v^*) + \partial_u \mathscr{I}[V](x; u^*, v^*) \;=\; 0, \qquad \partial_v H(x, \nabla V, u^*, v^*) + \partial_v \mathscr{I}[V](x; u^*, v^*) \;=\; 0. \tag{2}$$

The non-local term is the integral against the Lévy measure of the impact of control-perturbed jumps on the value function.

# Main results (3/3)

- **Worked example: drift-and-intensity game.** For the **central-bank-versus-speculator** game in which the central bank controls the drift $\mu \in [-\bar{\mu}, \bar{\mu}]$ and the speculator controls the up-jump intensity $\lambda_+ \in [\lambda_-, \lambda_{\max}]$, the saddle pair admits an explicit closed form in the positive Wiener–Hopf root $r_+$ of $\psi(r) = q$ from (Mészáros et al. 2023)
- **Numerical verification.** Algorithm A.1 (Howard policy iteration on the discretised HJI) converges geometrically to the closed-form saddle of Claim 4, at rate $\rho(q, \lambda) = \lambda/(q + \lambda)$ matching the contraction rate of the underlying Banach iteration.

# Main result 1 (1/2)

- We now fix the notation inherited — in full — from (Mészáros et al., 2023, §2): the script symbols $\mathscr{L}$, $\mathscr{R}_q$, $\mathscr{I}$ denote the stochastic generator, the resolvent, and the non-local operator respectively, whilst $\mathscr{U}$ and $\mathscr{V}$ denote the admissible strategy classes for Players I and II.

## Main result 1 (2/2)

- The blackboard symbols $\mathbb{E}$, $\mathbb{P}$, $\mathbb{R}$ carry their standard probabilistic and analytic meanings; furthermore, partial derivatives are written $\partial_x$ rather than in subscript form.
- These conventions are adopted without modification throughout — ensuring full consistency with the operator-theoretic framework established in the cited reference.

# Main result 2 (1/2)

- **Theorem 4.1 (Value coincidence).** Under **(A1)–(A3)**, Lipschitz running cost, and the Isaacs condition, we have $V^+(x) = V^-(x) =: V(x)$ for all $x \in \mathbb{R}$ — and this common function is the unique viscosity solution of the saddle HJI equation $qV(x) = H^*(x, \partial_x V, \partial_{xx} V) + \mathscr{I}^*[V](x)$.

# Main result 2 (2/2)

- **Three-step proof.** One sees that $V^+$ is a viscosity sub-solution of the upper HJI and $V^-$ a super-solution of the lower; the non-local operator $\mathscr{I}$ enters via the Pham lemma (Lem. 4.2), which controls small-jump truncation under **(A2)** precisely as the diffusive term does.
- **Identification.** The Isaacs condition collapses both HJI equations to a single operator; the comparison principle of (Mészáros et al., 2023, Thm 5.1) then yields $V^+ \leq V^-$ — and since $V^- \leq V^+$ holds always, equality follows.

# Main result 3 (1/2)

- We now show that $(u^*, v^*) \in \mathscr{U} \times \mathscr{V}$ is a saddle pair if and only if, at every interior point of the continuation region, the first-order stationarity condition vanishes for each player — with the non-local operator $\mathscr{I}[V]$ entering the $v$-player's condition alongside the Hamiltonian.

# Main result 3 (2/2)

- The $v$-condition is non-local in the Lévy sense: differentiating $\mathscr{I}[V](x;\, v) = (1+v)\widetilde{I}[V](x)$ with respect to $v$ yields $\widetilde{I}[V](x) = \int \big(V(x+z) - V(x) - z\,\partial_x V(x)\,\mathbb{1}_{|z|\le 1}\big)\nu(dz)$ — a global functional of $V$ over the full support of $\nu$.
- In the Brownian limit $\nu \equiv 0$, one has $\widetilde{I}[V] \equiv 0$ and the non-local condition collapses to $\partial_v H(x, \partial_x V, \partial_{xx} V, u^*, v^*) = 0$; the jump measure $\nu$ is thus precisely what distinguishes the Lévy saddle from the diffusive one.

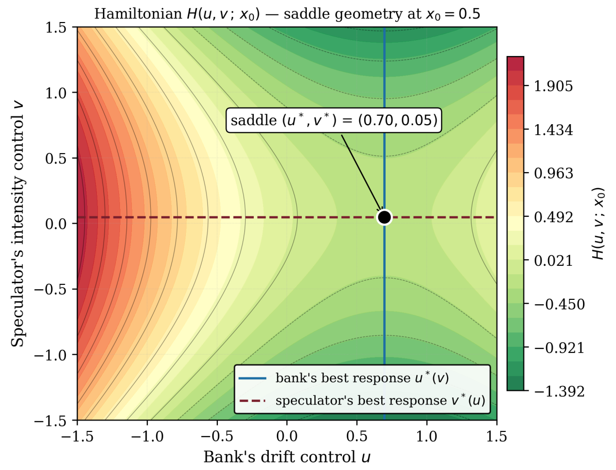
# Worked example (1/2)

- **Players and state.** We specialise the abstract framework to a zero-sum game — the central bank (Player I, $u \in \mathscr{U} = [-\bar{u}, \bar{u}]$) defends against a speculator (Player II, $v \in \mathscr{V} = [0, \bar{v}]$) — where the state $X = \log S$ follows a Kou jump-diffusion with Lévy triplet $(b, \sigma_0^2, \nu)$.

# Worked example (2/2)

- **Running cost.** Observe that the linear-quadratic payoff $\ell(x, u, v) = \alpha x + \frac{1}{2}\gamma u^2 - \frac{1}{2}\delta v^2$ penalises depreciation linearly, defence effort quadratically, and rewards attack effort quadratically — a zero-sum structure throughout.
- **Saddle mechanism.** The attacker's control perturbs the up-jump intensity of $X$; the Hamiltonian separates in $(u, v)$, and the saddle closes in terms of the positive Wiener–Hopf root $r_+ \in (0, \eta_+)$ satisfying $\psi^{0,0}(r_+) = q$.

# Numerical verification



## What we showed (1/2)

- We formulated a zero-sum stochastic differential game in which the driving noise is of Lévy type — thus allowing for jump discontinuities in the state dynamics alongside the diffusion component.
- The competing strategy classes $\mathscr{U}$ and $\mathscr{V}$ of the two players define the upper and lower value functions of the game.

# What we showed (2/2)

- The stochastic generator $\mathscr{L}$, under Lévy noise, acquires a non-local character — becoming an integro-differential operator that encodes the full jump structure.
- The Hamilton–Jacobi–Isaacs equation emerges as the governing equation for the game's value function.
- One sees, moreover, that the interplay between Lévy noise and the game-theoretic structure is directly reflected in the non-local form of this equation.

# Thank you — questions welcome

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