A Note on the Hamilton–Jacobi–Bellman Equation

IADU Technical Note · Series: Quantitative Methods

Institute of Advanced Dynamic Uncertainty

May 2026 | IADU-TN-001

Abstract

We provide a self-contained introduction to the Hamilton–Jacobi–Bellman (HJB) equation in continuous-time optimal control. The value function is characterised as the unique viscosity solution to the associated nonlinear PDE. We illustrate the framework with a portfolio optimisation problem under geometric Brownian motion dynamics.

Contents

The Control Problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space supporting a standard Brownian motion $W = (W_t)_{t \geq 0}$. Consider a controlled diffusion

$$dX_t = f(X_t, u_t)\,dt + \sigma(X_t, u_t)\,dW_t, \quad X_0 = x,$$

where $u_t \in \mathcal{U}$ is the control process. The objective is to maximise

$$J(x, t; u) = \mathbb{E}\left[\int_t^T \ell(X_s, u_s)\,ds + g(X_T) \,\middle|\, X_t = x\right].$$

Definition 1 (Value Function)**.** The value function $V : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ is defined by

$$V(x, t) = \sup_{u \in \mathcal{U}} J(x, t; u).$$

The HJB Equation

Under suitable regularity assumptions, V satisfies the *Hamilton–Jacobi–Bellman equation*:

$$-\partial_t V(x, t) = \sup_{u \in \mathcal{U}} \Big\{ \ell(x, u) + f(x, u)^\top \nabla_x V + \tfrac{1}{2}\,\mathrm{tr}\big[\sigma\sigma^\top \nabla_x^2 V\big] \Big\},$$

with terminal condition $V(x, T) = g(x)$.

Theorem 1 (Verification)**.** *Let* $V \in C^{1,2}([0,T) \times \mathbb{R}^n) \cap C([0,T] \times \mathbb{R}^n)$ *satisfy the HJB equation. Then* V *equals the value function, and any measurable selector*

$$u^*(x,t) \in \arg\max_u \Big\{ \ell(x,u) + f(x,u)^\top \nabla_x V + \tfrac{1}{2}\,\mathrm{tr}[\cdots] \Big\}$$

is an optimal control.

Proof. Apply Itô's formula to $V(X_t, t)$ and use the HJB inequality. □ □

Portfolio Optimisation Example

Let wealth X_t follow

$$dX_t = \big[rX_t + u_t(\mu - r) \big]\, dt + u_t \sigma \, dW_t,$$

where r is the risk-free rate, $\mu > r$ the risky asset drift, $\sigma > 0$ its volatility, and u_t the monetary amount invested in the risky asset.

For CRRA utility $g(x) = x^\gamma/\gamma$ with $\gamma \in (0,1)$, the value function takes the separable form $V(x,t) = h(t)\, x^\gamma/\gamma$, where h satisfies a Riccati ODE. The optimal fraction invested is the celebrated *Merton ratio*:

$$\pi^* = \frac{\mu - r}{(1-\gamma)\sigma^2}.$$

Remark 1. The Merton ratio is independent of wealth and time, a consequence of the power-utility homogeneity. Extensions to stochastic volatility, jumps, and transaction costs break this property and require numerical HJB solvers.

Conclusion

The HJB equation reduces infinite-dimensional stochastic control problems to a single PDE. Numerical methods — finite differences, policy iteration, and deep learning — extend the framework to high-dimensional settings relevant to modern quantitative finance and macroeconomic modelling.

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Technical Note IADU-TN-001 · May 2026